A Generalization of the Perfect Graph Theorem under the Disjunctive Index

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Abstract

In this paper we relate antiblocker duality between polyhedra, graph theory and the disjunctive procedure. In particular, we analyze the behavior of the disjunctive procedure over the clique relaxation, $\mathcal{K}(G)$, of the stable set polytope in a graph G and the one associated to its complementary graph, $\mathcal{K}(\bar{G})$. We obtain a generalization of the Perfect Graph Theorem proving that the disjunctive indices of $\mathcal{K}(G)$ and $\mathcal{K}(\bar{G})$ always coincide.

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1 Introduction

In this paper we relate antiblocker polyhedra duality as defined by Fulkerson in [5], graph theory and the sequential tightening procedure of Balas, Ceria and Cornuéjols [1]. These relationships will lead us to a generalization of the Perfect Graph Theorem [6].

Given a graph $G = (V, E)$, if $\omega(G)$ denotes the size of the largest clique and $\chi(G)$ its chromatic number, it is clear that $\chi(G) \geq \omega(G)$. If equality holds for G and every node induced subgraph G' of G, i.e. if $\chi(G') = \omega(G')$, the graph G is said to be *perfect*.

Berge conjectured [2] and Lovász proved [6] that, if a graph G is perfect then its complement, $\bar{G}$, is also perfect, a result known as the Perfect Graph Theorem.

On the other hand, Chvátal [4] established relationships between perfect graphs and polyhedral theory: defining

$$\mathcal{K}(G) = \{x \in \mathbb{R}_+^{|V|} : \sum_{i \in k} x_i \leq 1,\ k \text{ clique in } G\},$$

it is easy to see that any $0-1$ point in $\mathcal{K}(G)$ is the incidence vector of a stable set in G. Thus, the polytope $\mathcal{K}(G)$ is called the *clique relaxation* of the stable set polytope.

In [4], using Lovász's perfect graph theorem, Chvátal proved that a graph G is perfect if and only if the polytope $\mathcal{K}(G)$ has only integral vertices.

When the graph G is not perfect, it makes sense to look for tightening procedures for finding the convex hull of integer points in $\mathcal{K}(G)$. In this paper we work with the *disjunctive procedure*, a lift and project method developed by Balas, Ceria and Cornuéjols in [1], defined on polytopes of the form

$$\mathcal{K} = \{x \in \mathbb{R}^n_+ : Ax \le b,\ x_i \le 1,\ \text{for } i = 1, \dots, n\} = \{x \in \mathbb{R}^n : \tilde{A}x \le \tilde{b}\}.$$

This procedure can be briefly described as follows:

For fixed j, $1 \le j \le n$, the inequalities $\tilde{A}x \le \tilde{b}$ are multiplied by x_j and $1 - x_j$, obtaining a system of, in general, nonlinear inequalities. Then, x_j^2 is replaced by x_j and products of the form $x_i x_j$ are replaced by new variables y_i for $i \ne j$, obtaining a system of linear inequalities in the variables x and y. The polytope $M_j(\mathcal{K})$, defined by this system of linear inequalities, is projected back onto the x-space, by eliminating the y variables, and the resulting polytope is denoted by $P_j(\mathcal{K})$.

If $\operatorname{conv}(U)$ is the convex hull of the elements in $U \subset \mathbb{R}^n$ and $U^* = \operatorname{conv}(U \cap \mathbb{Z}^n)$, the following result, proved in [1], gives an alternative definition of the disjunctive procedure, much more geometrical in nature, and central to our discussion.

1.1 Theorem. *For any $j \in \{1, \dots, n\}$,*

$$P_j(\mathcal{K}) = \operatorname{conv}\bigl(\{x \in \mathcal{K} : x_j \in \{0, 1\}\}\bigr).$$

In particular, $\mathcal{K}^* \subset P_j(\mathcal{K}) \subset \mathcal{K}$.

Given $F = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ and defining

$$P_F(\mathcal{K}) = \operatorname{conv}(\{x \in \mathcal{K} : x_i \in \{0, 1\} \text{ for all } i \in F\}),$$

it was also proved in [1] that

$$P_F(\mathcal{K}) = P_{i_1}(P_{i_2}(\dots(P_{i_k}(\mathcal{K})))),$$

and in particular,

$$P_{\{1,\ldots,n\}}(\mathcal{K}) = \mathcal{K}^*.$$

This last result allows the definition of the *disjunctive index* of $\mathcal{K}$ as the minimum number of iterations needed in order to find the convex hull of the integer points in $\mathcal{K}$. In particular, if $\mathcal{K}$ is an integral polyhedron, the disjunctive index is zero.

Under these definitions, the Perfect Graph Theorem together with Chvátal's result, says that the disjunctive index of $\mathcal{K}(G)$ is zero if and only if the disjunctive index of $\mathcal{K}\left(\bar{G}\right)$ is zero.

On the other hand, a graph is *minimally imperfect* when it is not perfect but every node induced subgraph is perfect. If G is minimally imperfect, its complement $\bar{G}$ also is minimally imperfect, and it is not hard to prove that $\mathcal{K}(G)$ and $\mathcal{K}(\bar{G})$ have disjunctive index one.

From the previous remarks, the disjunctive index can be seen as an *imperfection* index of the graph G, and the main goal of the paper is to generalize the relationship between imperfection indices of a graph and its complement, in the following sense:

1.2 Theorem. *Given a graph G and its complement $\bar{G}$, their corresponding clique relaxations $\mathcal{K}(G)$ and $\mathcal{K}\left(\bar{G}\right)$ have the same disjunctive index.*

This theorem will be a consequence of a stronger result relating *antiblocking duality* and the disjunctive procedure.

Given $\mathcal{K} \subset \mathbb{R}^n_+$, $\mathcal{K}$ is an *antiblocking type* polyhedron if $\mathcal{K} = \{x \in \mathbb{R}^n_+ : Ax \leq 1\}$ for a matrix A with nonnegative entries and no zero columns.

Denoting by $\{e^1, ..., e^n\}$ the canonical basis of $\mathbb{R}^n$, it is not difficult to prove that $\mathcal{K} \subset \mathbb{R}^n_+$ is an antiblocking type polyhedron if and only if for every $x \in \mathcal{K}$ and $i = 1, ..., n$, $\left(x - x_i e^i\right) \in \mathcal{K}$.

Recalling that the *polar* of a polyhedron $\mathcal{K}$ in $\mathbb{R}^n$ is

$$\Pi(\mathcal{K}) = \{(\pi, \pi_0) \in \mathbb{R}^n \times \mathbb{R} : \pi x \leq \pi_0 \text{ for all } x \in \mathcal{K}\},$$

we define the *positive 1-polar* of $\mathcal{K}$ by

$$\Pi^1_+(\mathcal{K}) = \{\pi \in \mathbb{R}^n_+ : (\pi, 1) \in \Pi(\mathcal{K})\}.$$

If $\mathcal{K}$ is an antiblocking type polyhedron, its positive 1-polar is called the *antiblocker*, and is denoted by $\mathcal{K}^C$. It can be shown in this case that if B is the matrix whose rows are the extreme points of $\mathcal{K}$,

$$\mathcal{K}^C = \{\pi \in \mathbb{R}^n_+ : B\pi \leq 1\}, \tag{1.1}$$

so that $\mathcal{K}^C$ is also an antiblocking type polyhedron, and $\left(\mathcal{K}^C\right)^C = \mathcal{K}$, allowing us to refer to K and $\mathcal{K}^C$ as an antiblocking pair of polyhedra (see [5]).

Since $\mathcal{K}(G)$ is an antiblocking type polyhedron, and stable sets in $\bar{G}$ are cliques in G, by (1.1), $\left(\mathcal{K}\left(\bar{G}\right)\right)^*$ and $\mathcal{K}(G)$ define an antiblocking pair of polyhedra. Interchanging the roles of G and $\bar{G}$, $(\mathcal{K}(G))^*$ and $\mathcal{K}\left(\bar{G}\right)$ also define an antiblocking pair of polyhedra, and we can summarize these relationships by the following diagram

$$\begin{array}{ccc} \mathcal{K}(G) & \longleftrightarrow & \left(\mathcal{K}\left(\bar{G}\right)\right)^* \\ & \text{antiblocker} & \\ \text{convex hull} \downarrow & & \uparrow \text{ convex hull} \\ & & \\ (\mathcal{K}(G))^* & \longleftrightarrow & \mathcal{K}\left(\bar{G}\right) \\ & \text{antiblocker} & \end{array} \tag{1.2}$$

Let us now state the first simple result connecting antiblocking duality and the disjunctive procedure.

1.3 Lemma. *If $\mathcal{K}$ is an antiblocking type polytope with vertices in $[0,1]^n$ and $F \subset \{1,...,n\}$, then $P_F(\mathcal{K})$ is also an antiblocking type polytope.*

Proof. Clearly, we only need to prove that for any $j \in F$, $P_j(\mathcal{K})$ is an antiblocking type polytope. Recall that

$$P_j(\mathcal{K}) = \text{conv}\big(\{x \in \mathcal{K} : x_j \in \{0,1\}\}\big).$$

If $\{e^1,...,e^n\}$ is the canonical basis of $\mathbb{R}^n$, we will prove that for all $x \in P_j(\mathcal{K})$ and $i = 1,...,n$, $\left(x - x_i e^i\right) \in P_j(\mathcal{K})$. It obviously holds if $i = j$.

For the case $i \neq j$, let $x \in P_j(\mathcal{K})$ and $x^0 \in \{x \in \mathcal{K} : x_j = 0\}$, $x^1 \in \{x \in \mathcal{K} : x_j = 1\}$ such that

$$x = x_j x^1 + (1 - x_j)x^0.$$

Therefore

$$x - x_i e^i = x_j(x^1 - x_i^1 e^i) + (1 - x_j)(x^0 - x_i^0 e^i).$$

Since $\mathcal{K}$ is an antiblocking type polyhedron and $i \neq j$, $(x^1 - x_i^1 e^i) \in \{x \in \mathcal{K} : x_j = 1\}$ and $(x^0 - x_i^0 e^i) \in \{x \in \mathcal{K} : x_j = 0\}$. Then $\left(x - x_i e^i\right) \in P_j(\mathcal{K})$. □

So it makes sense to analyze $[P_F(\mathcal{K}(G))]^C$. One of the strongest results of the paper can be seen as an extension of Diagram 1.2, as follows:

$$\begin{array}{ccc}
\mathcal{K}(G) & \longleftrightarrow & [\mathcal{K}(G)]^C \\
 & \text{antiblocker} & \\
P_F \downarrow & & \uparrow P_F \\
 & & \\
P_F(\mathcal{K}(G)) & \longleftrightarrow & [P_F(\mathcal{K}(G))]^C \\
 & \text{antiblocker} & \\
 & & \\
\downarrow & & \uparrow \\
(\mathcal{K}(G))^* & \longleftrightarrow & \mathcal{K}\left(\bar{G}\right) \\
 & \text{antiblocker} &
\end{array}$$

More precisely, in Section 3 we will prove the following

1.4 Theorem. *If $\mathcal{K}(G)$ is the clique relaxation of the stable set polytope in a graph $G = (V, E)$ then for any $F \subset V$,*

$$P_F\left([P_F(\mathcal{K}(G))]^C\right) = [\mathcal{K}(G)]^C .$$

The proof will be based on the behavior of a single application of the disjunctive procedure, that is, when $F = \{j\}$ for any j. This first step is analyzed in the following section.

2 The antiblocker of $P_j(\mathcal{K}(G))$

Let us consider again a graph $G = (V, E)$, where $V = \{1, ..., n\}$ and $\mathcal{K}(G)$ is the clique relaxation of the stable set polytope. We will prove that, for any j, the following diagram holds

$$\begin{array}{ccc} \mathcal{K}(G) & \longleftrightarrow & [\mathcal{K}(G)]^C \\ & \text{antiblocker} & \\ P_j \downarrow & & \uparrow P_j \\ P_j(\mathcal{K}(G)) & \longleftrightarrow & [P_j(\mathcal{K}(G))]^C \\ & \text{antiblocker} & \end{array}$$

The key for proving this result is the characterization of valid inequalities of $P_j(\mathcal{K}(G))$ given in [3]. In order to keep the paper self-contained, we provide below the derivation of these inequalities.

Recalling that, for any $j \in V$, $P_j(\mathcal{K}(G))$ is the projection onto de x-space of the polyhedron $M_j(\mathcal{K}(G))$ that lies on a higher dimensional space, following the description of the disjunctive procedure given in Section 1 with $\mathcal{K} = \mathcal{K}(G)$, we see that $M_j(\mathcal{K}(G))$ is described by the system

$$\begin{array}{rcll} \sum_{i\in k} x_i + x_j - 1 \leq & \sum_{i\in k} y_i & \leq x_j & \forall k \in Q, \\ 0 \leq & y_i & \leq x_i & \forall i \in V \setminus \{j\}, \\ 0 \leq & x_j & = y_j. & \end{array}$$

where Q denotes the set of maximal cliques in the graph G.

Let $\Gamma(j) = \{i \in V : [i, j] \in E\}$, $V' = V \setminus (\Gamma(j) \cup \{j\})$, and let Q' be the set of all cliques in Q that do not contain a given $j \in V$. Working over the previous system, we can see that given $x \in \mathcal{K}(G)$, $x \in P_j(\mathcal{K}(G))$ if and only if there exists $y \in \mathbb{R}^{|V'|}$ such that

$$\begin{aligned} \sum_{i\in k} x_i + x_j - 1 \leq \sum_{i\in k\cap V'} y_i &\leq x_j, \qquad k \in Q', \\ 0 \leq y_i &\leq x_i, \qquad i \in V', \end{aligned}$$

or equivalently, if the system

$$\begin{aligned} \sum_{i\in k\cap V'} y_i + z_k &= x_j, && k \in Q', \\ 0 \leq y_i &\leq x_i, && i \in V', \\ 0 \leq z_k &\leq 1 - \sum_{i\in k} x_i, && k \in Q' \end{aligned}$$

is feasible. If so, by Farkas' lemma the system

$$\begin{aligned} -\sum_{k/i\in k} u_k + v_i \geq 0, &\quad i \in V' \\ -u_k + w_k \geq 0 &\quad k \in Q' \\ v, w \geq 0 & \\ -\left(\sum_{k\in Q'} u_k\right) x_j + \sum_{i\in V'} v_i x_i + \sum_{k\in Q'} w_k \left(1 - \sum_{i\in k} x_i\right) < 0 & \end{aligned} \tag{2.1}$$

should be infeasible.

It is easy to see that (2.1) is infeasible if and only if there is no $u \in \mathbb{R}^{|Q'|}$ such that

$$-\left(\sum_{k\in Q'} u_k\right) x_j + \sum_{i\in V'} x_i \max(0, \sum_{k/i\in k} u_k) + \sum_{k\in Q'} \max(0, u_k) \left(1 - \sum_{i\in k} x_i\right) < 0.$$

In other words, given $x \in \mathcal{K}(G)$, $x \in P_j(\mathcal{K}(G))$ if and only if, for every $u \in \mathbb{R}^{|Q'|}$,

$$\begin{aligned} \left(\sum_{k\in Q'} u_k\right) x_j - \sum_{i\notin\Gamma(j)} x_i \max(0, \sum_{k/i\in k} u_k) + \sum_{k\in Q'} \sum_{i\in k} \max(0, u_k)\, x_i \\ \leq \sum_{k\in Q'} \max(0, u_k). \end{aligned} \tag{2.2}$$

Let us observe that each $u \in \mathbb{R}^{|Q'|}$ defines a partition of Q' given by

$$P = \{i \in Q' : u_i > 0\} \quad \text{and} \quad \bar{P} = Q' \setminus P = \{i \in Q' : u_i \leq 0\}.$$

Redefining u_i as $(-u_i)$ for all $i \in \bar{P}$, (2.2) becomes

$$\left(\sum_{k \in P} u_k - \sum_{k \in \bar{P}} u_k \right) x_j + \sum_{\substack{i \in V' \\ \sum\limits_{\substack{i \in k \\ k \in P}} u_k > \sum\limits_{\substack{i \in k \\ k \in \bar{P}}} u_k}} x_i \Big(\sum_{\substack{k / i \in k \\ k \in \bar{P}}} u_k - \sum_{\substack{k / i \in k \\ k \in P}} u_k \Big) + \sum_{k \in P} \sum_{i \in k} u_k x_i$$
$$\leq \sum_{k \in P} u_k$$

or equivalently,

$$\left(\sum_{k \in P} u_k - \sum_{k \in \bar{P}} u_k \right) x_j + \sum_{i \in \Gamma(j)} \Big(\sum_{\substack{k / i \in k \\ k \in P}} u_k \Big) x_i + \sum_{i \in V'} \min\Big(\sum_{\substack{k / i \in k \\ k \in P}} u_k, \sum_{\substack{k / i \in k \\ k \in \bar{P}}} u_k \Big) x_i$$
$$\leq \sum_{k \in P} u_k.$$

Therefore, the following theorem is proved

2.1 Theorem ([3]). *Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$, and let $j \in V$. If $x \in \mathcal{K}(G)$, then $x \in P_j(\mathcal{K}(G))$ if and only if*

$$\left(\sum_{k \in P} u_k - \sum_{k \in \bar{P}} u_k \right) x_j + \sum_{i \in \Gamma(j)} \Big(\sum_{\substack{k / i \in k \\ k \in P}} u_k \Big) x_i + \sum_{i \in V'} \min\Big(\sum_{\substack{k / i \in k \\ k \in P}} u_k, \sum_{\substack{k / i \in k \\ k \in \bar{P}}} u_k \Big) x_i$$
$$\leq \sum_{k \in P} u_k$$

for every $P \subset Q'$ and $u \in \mathbb{R}_+^{|Q'|}$, where Q' is the set of all the maximal cliques in G not containing j and $\bar{P} = Q' \setminus P$.

Let us now prove the following

2.2 Theorem. *If $\mathcal{K}(G)$ is the clique relaxation of the stable set polytope in a graph $G=(V,E)$, then for any $j \in V$*

$$P_j\left(\left[P_j\left(\mathcal{K}(G)\right)\right]^C\right) = \left[\mathcal{K}(G)\right]^C.$$

Proof. Since $P_j(\mathcal{K}(G)) \subset \mathcal{K}(G)$, then $\left[P_j(\mathcal{K}(G))\right]^C \supset \left[\mathcal{K}(G)\right]^C$. On the other hand, by the relationship shown in Diagram 1.2, $\mathcal{K}(G)$ and $\mathcal{K}\left(\bar{G}\right)^*$ define an antiblocking pair of polyhedra. Then $\left[\mathcal{K}(G)\right]^C$ is an integral polyhedron and

$$P_j\left(\left[P_j\left(\mathcal{K}(G)\right)\right]^C\right) \supset P_j\left(\left[\mathcal{K}(G)\right]^C\right) = \left[\mathcal{K}(G)\right]^C.$$

Let us now prove that

$$P_j\left(\left[P_j\left(\mathcal{K}(G)\right)\right]^C\right) \subset \left[\mathcal{K}(G)\right]^C.$$

For this purpose, we only need to verify that every valid inequality for $P_j(\mathcal{K}(G))$ of the form $\gamma x \leq 1$ with $\gamma_j \in \{0,1\}$ is a valid inequality for $\mathcal{K}(G)$.

Following the notation of Theorem 2.1, we only have to analyze inequalities of the form

$$\gamma x = \left(1 - \sum_{k \in \bar{P}} u_k\right) x_j + \sum_{i \in \Gamma(j)} \Big(\sum_{\substack{k/i \in k \\ k \in P}} u_k\Big) x_i + \sum_{i \in V'} \min\Big(\sum_{\substack{k/i \in k \\ k \in P}} u_k, \sum_{\substack{k/i \in k \\ k \in \bar{P}}} u_k\Big) x_i \leq 1$$

where $P \subset Q'$ and $u \in \mathbb{R}_+^{|Q'|}$ such that $\sum_{k \in P} u_k = 1$.

If $\gamma_j = 0$ and $x \in \mathcal{K}(G)$ then

$$\begin{aligned}
\gamma x &= \sum_{i \in \Gamma(j)} \Big(\sum_{\substack{k/i \in k \\ k \in P}} u_k\Big) x_i + \sum_{i \in V'} \min\Big(\sum_{\substack{k/i \in k \\ k \in P}} u_k, \sum_{\substack{k/i \in k \\ k \in \bar{P}}} u_k\Big) x_i \\
&\leq \sum_{i \in \Gamma(j)} \Big(\sum_{\substack{k/i \in k \\ k \in P}} u_k\Big) x_i + \sum_{i \in V'} \sum_{\substack{k/i \in k \\ k \in P}} u_k x_i \\
&= \sum_{i \in V \setminus \{j\}} \sum_{\substack{k/i \in k \\ k \in P}} u_k x_i = \sum_{k \in P} u_k \sum_{i \in k} x_i \leq 1.
\end{aligned}$$

Now, if $\gamma_j = 1$ then $\sum_{k\in\bar{P}} u_k = 0$, and $u_k = 0$ for every $k \in \bar{P}$. In this case, if $x \in \mathcal{K}(G)$ we have

$$\begin{aligned}
\gamma x &= x_j + \sum_{i\in\Gamma(j)} (\sum_{\substack{k/i\in k\\ k\in P}} u_k) x_i + \sum_{i\in V'} \min(\sum_{\substack{k/i\in k\\ k\in P}} u_k, 0) x_i \\
&= x_j + \sum_{i\in\Gamma(j)} (\sum_{\substack{k/i\in k\\ k\in P}} u_k) x_i \\
&= x_j + \sum_{k\in P} u_k \sum_{i\in\Gamma(j)\cap k} x_i \\
&= \sum_{k\in P} u_k \left(x_j + \sum_{i\in\Gamma(j)\cap k} x_i \right) \leq 1. \qquad \square
\end{aligned}$$

3 The Disjunctive Index of $\mathcal{K}(G)$ and $\mathcal{K}(\bar{G})$

At this point, it is natural to ask whether given a graph $G = (V, E)$ and any $F \subset V = \{1, ..., n\}$, the following diagram holds

$$\begin{array}{ccc}
\mathcal{K}(G) & \longleftrightarrow & (\mathcal{K}(G))^C \\
& \text{antiblocker} & \\
P_F \downarrow & & \uparrow P_F \\
& & \\
P_F(\mathcal{K}(G)) & \longleftrightarrow & (P_F(\mathcal{K}(G)))^C \\
& \text{antiblocker} &
\end{array}$$

Actually, it will be enough to see whether Theorem 2.2 is valid substituting $\mathcal{K}(G)$ for $P_F(\mathcal{K}(G))$, that is, whether

$$P_j \left([P_j (P_F (\mathcal{K}(G)))]^C \right) \subset [P_F (\mathcal{K}(G))]^C .$$

For this purpose, we state some more definitions and results.

For $H \subset V$ let us set $\Gamma(H) = \{j \in V : [i,j] \in E \text{ for some } i \in H\}$, and for fixed $F \subset V$ and any $H \subset F$ let $V_H = V \setminus (F \cup \Gamma(H))$.

Now if

$$\mathcal{K}_H = \{x \in \mathcal{K}(G) : x_i = 1 \text{ if } i \in H, \text{ and } x_i = 0 \text{ if } i \in F \setminus H\},$$

we have

$$P_F(\mathcal{K}(G)) = \operatorname{conv}\left(\bigcup_{H \subset F} \mathcal{K}_H\right)$$

and it is not difficult to see that

$$[P_F(\mathcal{K}(G))]^C = \bigcap_{H \subset F} \Pi_+^1(\mathcal{K}_H).$$

Clearly, $\mathcal{K}_H = \emptyset$ if H is not a stable set in G, and therefore, in what follows we restrict our attention to the case when H is stable.

Defining $x^H \in \mathbb{R}^{|V \setminus V_H|}$ by

$$x_i^H = \begin{cases} 1 & \text{if } i \in H, \\ 0 & \text{if } i \in (F \setminus H) \cup \Gamma(H), \end{cases}$$

we have $\mathcal{K}_H = \{x^H\}$ if $V_H = \emptyset$, whereas if $V_H \neq \emptyset$ and denoting by G_H the subgraph induced by V_H

$$\mathcal{K}_H = \{(x^H, x) \in \mathbb{R}^n : x \in \mathcal{K}(G_H)\}.$$

3.1 Lemma. *For every stable set H in G and $j \in V_H$,*

$$P_j\left(\Pi_+^1(P_j(\mathcal{K}_H))\right) \subset \Pi_+^1(\mathcal{K}_H). \tag{3.1}$$

Proof. If $V_H = \emptyset$, then $\mathcal{K}_H = \{x^H\}$ and $P_j(\mathcal{K}_H) = \mathcal{K}_H$, so that we have

$$\Pi^1_+ (P_j(\mathcal{K}_H)) = \Pi^1_+ (\mathcal{K}_H) \tag{3.2}$$

and the result follows since

$$P_j \left(\Pi^1_+ (\mathcal{K}_H)\right) \subset \Pi^1_+ (\mathcal{K}_H). \tag{3.3}$$

On the other hand, if $V_H \neq \emptyset$ for any $j \in V_H$ we must have,

$$P_j(\mathcal{K}_H) = \{(x^H, x) \in \mathbb{R}^n : x \in P_j(\mathcal{K}(G_H))\},$$

and $(\pi^1, \pi^2) \in \Pi^1_+ (P_j(\mathcal{K}_H))$ if and only if

$$\sum_{i \in H} \pi^1_i + \pi^2 x \leq 1 \quad \text{for every} \quad x \in P_j(\mathcal{K}(G_H)),$$

so that $\pi^2 \in [P_j(\mathcal{K}(G_H))]^C$.

We would like to prove now that if $\pi^2_j \in \{0, 1\}$ then

$$\sum_{i \in H} \pi^1_i + \pi^2 x \leq 1$$

for every $x \in \mathcal{K}(G_H)$. But if $(\pi^1, \pi^2) \in \Pi^1_+ (P_j(\mathcal{K}_H))$ is such that $\pi^2_j \in \{0, 1\}$, we must have

$$\pi^2 \in P_j \left([P_j(\mathcal{K}(G_H))]^C \right)$$

and applying Theorem 2.2 to $\mathcal{K}(G_H)$, we conclude that $\pi^2 \in [\mathcal{K}(G_H)]^C$.

If $\pi^2_j = 1$, since $e^j \in P_j(\mathcal{K}(G_H))$,

$$\sum_{i \in H} \pi^1_i + \pi^2 e^j = \sum_{i \in H} \pi^1_i + 1 \leq 1$$

and then $\sum_{i \in H} \pi^1_i = 0$. Therefore, for every $x \in \mathcal{K}(G_H)$, since $\pi^2 \in [P_j(\mathcal{K}(G_H))]^C$,

$$\sum_{i \in H} \pi^1_i + \pi^2 x = \pi^2 x \leq 1.$$

If $\pi_j^2 = 0$,

$$\sum_{i\in H} \pi_i^1 + \pi^2 x = \sum_{i\in H} \pi_i^1 + \pi^2 \left(x - x_j e^j\right).$$

Since for any $x \in \mathcal{K}(G_H)$, $\left(x - x_j e^j\right) \in P_j\left(\mathcal{K}(G_H)\right)$, we have

$$\sum_{i\in H} \pi_i^1 + \pi^2 x = \sum_{i\in H} \pi_i^1 + \pi^2 \left(x - x_j e^j\right) \leq 1. \qquad \square$$

Finally we are able to prove

3.2 Theorem. *If $\mathcal{K}(G)$ is the clique relaxation of the stable set polytope in a graph $G = (V, E)$ and $F \subset V$ then, for any $j \in V \setminus F$,*

$$P_j\left(\left[P_j\left(P_F\left(\mathcal{K}(G)\right)\right)\right]^C\right) \subset \left[P_F\left(\mathcal{K}(G)\right)\right]^C.$$

Proof. Using the notation of previous paragraphs,

$$P_F\left(\mathcal{K}(G)\right) = \operatorname{conv}\left(\bigcup_{H\subset F} \mathcal{K}_H\right),$$

and for any $j \in V \setminus F$,

$$P_j\left(P_F\left(\mathcal{K}(G)\right)\right) = \operatorname{conv}\left(\bigcup_{H\subset F} P_j\left(\mathcal{K}_H\right)\right)$$

and

$$\left[P_j\left(P_F\left(\mathcal{K}(G)\right)\right)\right]^C = \bigcap_{H\subset F} \Pi_+^1(P_j\left(\mathcal{K}_H\right)).$$

By the monotonicity of the disjunctive procedure we must have

$$P_j\left(\left[P_j\left(P_F\left(\mathcal{K}(G)\right)\right)\right]^C\right) = P_j\left(\bigcap_{H\subset F} \Pi_+^1\left(P_j\left(\mathcal{K}_H\right)\right)\right) \subset \bigcap_{H\subset F} P_j\left(\Pi_+^1\left(P_j\left(\mathcal{K}_H\right)\right)\right)$$

so that now by Lemma 3.1,

$$P_j\left(\left[P_j\left(P_F\left(\mathcal{K}(G)\right)\right)\right]^C\right) \subset \bigcap_{H\subset F} \Pi_+^1\left(\mathcal{K}_H\right) = \left[P_F\left(\mathcal{K}(G)\right)\right]^C. \qquad \square$$

The main result of the paper can be obtained as a corollary of the previous theorem.

3.3 Theorem. *If $\mathcal{K}(G)$ is the clique relaxation of the stable set polytope in a graph $G = (V, E)$ then, for any $F \subset V$,*

$$P_F\left([P_F(\mathcal{K}(G))]^C\right) = [\mathcal{K}(G)]^C .$$

Proof. Suppose $F \subset V$ is given. Following the same ideas of the proof of Theorem 2.2, we see that

$$P_F\left([P_F(\mathcal{K}(G))]^C\right) \supset [\mathcal{K}(G)]^C ,$$

and so we only need to prove

$$P_F\left([P_F(\mathcal{K}(G))]^C\right) \subset [\mathcal{K}(G)]^C .$$

If $F = \{i_1, ..., i_p\}$ with $p \geq 2$, then $P_F(\mathcal{K}(G)) = P_{i_1}\left(P_{F\setminus\{i_1\}}(\mathcal{K}(G))\right)$, and applying Theorem 3.2 to $P_{F\setminus\{i_1\}}(\mathcal{K}(G))$ we have

$$P_{i_1}\left([P_F(\mathcal{K}(G))]^C\right) \subset \left[P_{F\setminus\{i_1\}}(\mathcal{K}(G))\right]^C .$$

Finally, by the monotonicity of the disjunctive procedure and applying the same reasoning for $i_2, ..., i_p$, we obtain $P_F\left([P_F(\mathcal{K}(G))]^C\right) \subset [\mathcal{K}(G)]^C$. □

This result naturally leads to our generalization of the Perfect Graph Theorem.

3.4 Theorem (Generalized Perfect Graph Theorem). *Given a graph $G = (V, E)$ and its complement $\bar{G}$, if $P_F(\mathcal{K}(G)) = \mathcal{K}(G)^*$ for some $F \subset V$ then*

$P_F(\mathcal{K}(\bar{G})) = \mathcal{K}(\bar{G})^*$. *In particular,* $\mathcal{K}(G)$ *and* $\mathcal{K}(\bar{G})$ *have the same disjunctive index.*

Proof. By Diagram 1.2, $\mathcal{K}(\bar{G}) = \left[\mathcal{K}(G)^*\right]^C$, and thus $P_F\left(\mathcal{K}(\bar{G})\right) = P_F\left(\left[\mathcal{K}(G)^*\right]^C\right)$. Also, by hypothesis, $\mathcal{K}(G)^* = P_F(\mathcal{K}(G))$, so that

$$P_F\left(\mathcal{K}(\bar{G})\right) = P_F\left(\left[P_F(\mathcal{K}(G))\right]^C\right),$$

and we may apply Theorem 3.3 to obtain

$$P_F\left(\left[P_F(\mathcal{K}(G))\right]^C\right) = \left[\mathcal{K}(G)\right]^C.$$

Using again antiblocking duality between $\mathcal{K}(G)$ and $\mathcal{K}(\bar{G})^*$, we finally obtain $P_F(\mathcal{K}(\bar{G})) = \mathcal{K}(\bar{G})^*$. □

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