A polyhedral approach to the stability of a family of coalitions*

N. Aguilera[†] M. Escalante[‡]

August 5, 2008

Abstract

Using polyhedral combinatorics and graph theory results, we study unstable families of coalitions and analyze the effect of adding players or deleting coalitions in order to obtain a stable family. We also generalize the definitions and results to the case of fractional participation of players in coalitions.

Key words: cooperative games; family of coalitions; stability; polyhedral theory.

1 Introduction

In cooperative game theory, the coalitions are all the subsets of a set $N = \{1, 2, \ldots, n\}$ of players. Certainly, the possible number of coalitions is exponentially large, and it may not be feasible in practice to consider all of them, for instance, in order to evaluate the worth of each. It may even be the case that some of the players in a coalition may not get to meet or communicate with each other, so that some coalitions may not be actually formed. Therefore, it makes sense to restrict the attention to subfamilies of coalitions, eventually with significantly fewer elements than 2^n, and in this case identify properties which are decisive, whether to design a new family of coalitions or to modify an existing one.

When studying transferable utility games in characteristic function form, a function v is defined on the subsets of N. Often the condition $v(\emptyset) = 0$ is required, mostly for convenience. It is also natural in cooperative game theory to ask that v be *superadditive*, i.e.,

$$v(K') + v(K'') \leq v(K' \cup K'') \quad \text{for all } K', K'' \subset N \text{ with } K' \cap K'' = \emptyset. \tag{1.1}$$

However, unless explicitly stated, these conditions ($v(\emptyset) = 0$ and (1.1)) will not be assumed in what follows.

A fundamental concept in cooperative game theory is that of the core of a characteristic function v,

$$C(v) = \{x \in \mathbb{R}^n : x(N) \leq v(N) \text{ and } x(A) \geq v(A) \text{ for all } A \subset N, A \neq \emptyset\},$$

---

[*] This work is supported in part by CONICET (Argentina) Grant PIP 5810, ANPCyT (Argentina) Grant PICT 38035 and PICT 38036 .

[†] CONICET and Universidad Nacional del Litoral

[‡] CONICET and Universidad Nacional de Rosario

where for $A \subset N$, $A \neq \emptyset$, we write $x(A) = \sum_{i \in A} x_i$.

If $\mathcal{K}$ is a subfamily of coalitions, a natural generalization of the core concept is the $\mathcal{K}$-core, $C(v, \mathcal{K})$, the set of allocations acceptable to all coalitions in $\mathcal{K}$ according to v:

$$C(v, \mathcal{K}) = \{x \in \mathbb{R}^n : x(N) \leq v(N) \text{ and } x(K) \geq v(K) \text{ for all } K \in \mathcal{K}, K \neq \emptyset\}.$$

So as to avoid trivial technical exceptions, we will assume from now on that $\emptyset \notin \mathcal{K}$.

As is the case of the core, the $\mathcal{K}$-core $C(v, \mathcal{K})$ is a polyhedron, and it is useful to express it in matricial terms. To this end, let $M(\mathcal{K})$ be the matrix whose rows are indexed by the set of players and its columns are the incidence vectors of the coalitions in $\mathcal{K}$. Denoting by $v(\mathcal{K})$ the vector with entries $(v(\mathcal{K}))_K = v(K)$, $K \in \mathcal{K}$, we may write

$$C(v, \mathcal{K}) = \left\{x \in \mathbb{R}^n : x(N) \leq v(N) \text{ and } M(\mathcal{K})^T x \geq v(\mathcal{K})\right\}, \tag{1.2}$$

where for real vectors a, b of the same dimension m, the inequality $a \geq b$ means that $a_s \geq b_s$ for $s = 1, \ldots, m$.

As is well known, the possibility of finding an acceptable allocation to all coalitions in $\mathcal{K}$, i.e., $C(v, \mathcal{K}) \neq \emptyset$, may be written as a linear program:

$$z^*(v, \mathcal{K}) = \min\{\mathbf{1} \cdot x : M(\mathcal{K})^T x \geq v(\mathcal{K})\}, \tag{1.3}$$

where with $\mathbf{0}$ and $\mathbf{1}$ we denote the vectors whose components are all zeroes or all ones (not always with the same number of components), respectively.

We notice that the program (1.3) is always feasible (since $\emptyset \notin \mathcal{K}$ and therefore an appropriately large multiple of $\mathbf{1}$ is feasible), although it may be unbounded.

We arrive, then, to the fundamental relation

$$C(v, \mathcal{K}) \neq \emptyset \quad \text{if and only if} \quad z^*(v, \mathcal{K}) \leq v(N). \tag{1.4}$$

Linear programming theory leads to the dual program to that in (1.3),

$$\max\{v(\mathcal{K}) \cdot y : y \geq \mathbf{0}, M(\mathcal{K}) y = \mathbf{1}\}, \tag{1.5}$$

and we observe that the underlying polyhedron,

$$D(\mathcal{K}) = \{y \geq \mathbf{0} : M(\mathcal{K}) y = \mathbf{1}\},$$

is independent of v, i.e., it is intrinsic to $\mathcal{K}$, and therefore of crucial importance.

Provided there are no zero columns in $M(\mathcal{K})$, or equivalently, that $\emptyset \notin \mathcal{K}$—as we are assuming—$D(\mathcal{K})$ is always bounded, although it might be empty, which happens exactly when the program (1.3) is unbounded (since it is feasible), *regardless of the values of* v. More precisely,

Lemma 1.6. *The following are equivalent:*

- $D(\mathcal{K})$ *is empty,*
- $z^*(v, \mathcal{K}) = -\infty$ *for some characteristic function* v*,*
- $z^*(v, \mathcal{K}) = -\infty$ *for all characteristic functions* v*.*

This leads to a first classification of families of coalitions:

Definition 1.7. The family $\mathcal{K}$ is *balanced* if $D(\mathcal{K}) \neq \emptyset$. $\Diamond$

Thus, for any characteristic function v, when $\mathcal{K}$ is balanced the program (1.5) has a finite optimum, and therefore so does the program (1.3), and if $\mathcal{K}$ is unbalanced then $C(v, \mathcal{K}) \neq \emptyset$.

Since the entries of the matrix $M(\mathcal{K})$ are 0-1, the polytope $D(\mathcal{K})$ has a combinatorial meaning, a particular important case being when all its extreme points are integer, and combinatorial properties are reflected in properties of the family $\mathcal{K}$. Although the formal definition will be given later (in definition 2.5), let us say for the moment that $\mathcal{K}$ is *partitionable* if $D(\mathcal{K})$ has no fractional extreme points (see proposition 2.6). Or, in other words, $\mathcal{K}$ is partitionable if and only if either $D(\mathcal{K}) = \emptyset$ or $D(\mathcal{K})$ has only 0-1 extreme points.

Partitionability is defined in terms of the dual problem (1.5), but from the primal side, the important issue is to find conditions on $\mathcal{K}$ so that the $\mathcal{K}$-core is not empty for families of v's as large as possible, and this is referred to as the *stability* of $\mathcal{K}$.

Many kinds of stability for families of coalitions have been studied. For example, Boros, Gurvich and Vasin [4] considered v-stability, V-stability and g-stability, showing that these properties are equivalent under certain conditions. For instance, a family $\mathcal{K}$ is v-stable if $C(v, \mathcal{K}) \neq \emptyset$ for every superadditive characteristic function v. Examples of v-stable families of coalitions were considered by Gurvich and Vasin [12, 13], Kaneko and Wooders [15] and Quint [22], among others.

Boros et al. [4] and Kaneko and Wooders [15] related partitionability (i.e., the integrality of $D(\mathcal{K})$) and v-stability:

Theorem 1.8. *A family $\mathcal{K}$ is v-stable if and only if it is partitionable.*

After presenting some more definitions and background results in section 2, in section 3 we introduce a weaker type of stability, initially defined on a small subset of characteristic functions, but shown to hold for a larger family than those of superadditive functions, and prove results in the spirit of theorem 1.8, such as the equivalence between partitionability and stability.

Stability of a family of coalitions $\mathcal{K}$ is a desirable property, but what can be done if $\mathcal{K}$ is unstable? Some of the things that first come to mind in order to bring stability are to modify the game by adding or removing players or coalitions, while trying to keep as much as possible the properties of the original game.

In section 4 we show that we may always add a single player to some of the coalitions (without increasing the number of coalitions) in an unstable family $\mathcal{K}$ and obtain a stable but unbalanced family $\widetilde{\mathcal{K}}$. Since the $\widetilde{\mathcal{K}}$-core is unbounded (for any characteristic function), the directions will depend on the attributes of the added player, but not on the characteristic functions. We show examples of different kinds of added players, most prominently the *dummy* player, participating in no coalition, and the *mediator* who participates in every coalition. For the latter we show that the resulting family is balanced and stable if and only if the grand coalition, N, is initially in $\mathcal{K}$.

Of course, we can always single out a player from the original set N, and remove her from all coalitions (obtaining a dummy player) or add her to every coalition (obtaining a mediator), so that we obtain similar results without the explicit addition of any player.

The techniques for bringing stability used in section 4 imply that, for any characteristic function, there will always be allocations in the $\widetilde{K}$-core violating individual rationality.

Since individual rationality is a natural assumption in game theory, i.e., $x_i \geq v(\{i\})$ for any admissible payoff vector $x = (x_1, \ldots, x_n)$ and any $i \in N$, to enforce it we may ask that all singletons be elements of $\mathcal{K}$, which we may write as $[N] \subset \mathcal{K}$, where $[N]$ denotes the set of all the individual coalitions. In this case, the incidence vectors of the singletons (i.e., those for which the K-th component is 1 if $K = \{i\}$ for some $i \in N$) is always an element of $D(\mathcal{K})$, so $\mathcal{K}$ is balanced.

Recalling that a 0-1 matrix A with no zero columns is perfect if the polyhedron $\{y \in \mathbb{R}^d : Ay \leq \mathbf{1}, y \geq \mathbf{0}\}$ is integral, by introducing slack variables it follows that:

Theorem 1.9. *$\mathcal{K} \cup [N]$ is partitionable if and only if $M(\mathcal{K})$ is perfect.*

Le Breton, Owen and Weber [17] (and independently Kuipers [16] and Boros and Gurvich [3]), showed that the family of coalitions $\mathcal{K} \cup [N]$ is partitionable if and only if a corresponding hypergraph is normal. This result is seen to be equivalent to theorem 1.9, using results by Lovász [18] (relating normal hypergraphs and perfect graphs), and Chvátal [8] (relating perfect graphs and perfect matrices).

Thus, although the results of sections 3 and 4 depend mostly on linear programming theory, it is natural to use the tools of polyhedral combinatorics to study the partitionability of families of coalitions when $[N] \subset \mathcal{K}$. For the reader's convenience, we have included section 9 as a small appendix in which we give basic definitions and results from graph theory and polyhedral combinatorics.

Using Chvátal's characterization of perfect matrices [8] and the theory of antiblocker duality as developed by Fulkerson [10], we turn to the study of how to obtain a balanced and stable family, while trying to keep as much as possible the shape of the original N and $\mathcal{K}$. In section 5 we study the effect of adding several players in order to obtain a balanced and stable family, and in section 6 we study the deletion of coalitions. It is natural to think, then, of *instability indices* of a family $\mathcal{K}$ containing $[N]$: the minimum number of players to be added in order to make the family stable (in the case where the associated graph is perfect) and the minimum number of coalitions to be deleted in order to obtain a stable family.

Let us remark that we do not include in this study removal of players or addition of coalitions. In general, after removing a minimal set of players in order to obtain stability, many properties of the given family are not preserved. Moreover, the problem of finding such a set of players is *NP*-complete. Finally, using graph theoretical results it is not difficult to prove that the addition of coalitions to an unstable family never helps in finding a new stable family.

Motivated by our analysis of the addition of players in coalitions, in section 8 we extend definitions and results to study the case where players can participate in coalitions a fractional amount (as opposed to just 0 or 1).

Traditionally, in cooperative game theory each player either participates in a coalition or not. This can be interpreted as two levels of activity: active or inactive. A natural extension is to model the case in which players can contribute to the worth of a coalition at several levels of activity. Hsiao and Raghavan [14] introduced multi-choice games in which players have the same number of activity levels and with each level of activity a weight is associated in order to discriminate between those levels. More recently, other authors such as van den Nouweland et al. [19] and Peters and Zank [21], relaxed some of the assumptions, extending well-known concepts of traditional cooperative games to multichoice games.

In section 8 we present a slightly different approach for players with levels of participation in coalitions: we allow an infinite set of levels (but assume that only a few of all the possible coalitions are considered). This was first introduced in [2] as a fuzzy cooperation between players, that is participation at any level between noncooperation and full cooperation.

In [23], the authors introduce the concept of cores and stable sets for games with *fuzzy* coalitions. For convex fuzzy games they prove that all cores coincide and that the core is the unique stable set.

For more background on this subject we refer the reader to the book by Branzei et al. [5] and the references therein

In this paper, with a suitable extension of their definitions, stability and partitionability also coincide. Besides, we show that, if an unstable family of coalitions contains the individual coalitions, it can be made stable by the addition of players (with fractional participation) without asking for the individual rationality of the added players.

As a final comment, let us say that the design of families of stable coalitions is not a simple task. For instance, a naive possibility is to arrange the n players on a circle, and, choosing some $k < n/2$, decide that every subset of k consecutive players will form a coalition. Our results show that, even when adding the individual coalitions (i.e., $[N]$), it is not possible to obtain a balanced and stable family unless $k = 1$ or $k = 2$ and n is even.

2 Further definitions and background

In this section we present some notations which may be treated differently by other sources, and some results which we will need.

For $A \subset N$ and $i \in N$, $\chi(A, i)$ is the indicator function of A evaluated at i,

$$\chi(A,i) = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{otherwise,} \end{cases}$$

and the incidence vector of A in N is the vector $(\chi(A, 1), \chi(A, 2), \dots, \chi(A, n))$. $A \setminus B$ denotes the set difference of the sets A and B.

Definition 2.1. A *balanced weighting of* $\mathcal{K}$ is a function $w : \mathcal{K} \to \mathbb{Z}_+$, such that for some $m \in \mathbb{Z}$, $m > 0$,

$$\sum_{K \in \mathcal{K}} \chi(K, i)\, w(K) = m \quad \text{for every } i \in N. \tag{2.2}$$

In this case we say that w has *multiplicity* m, and its *support* is the subfamily $\mathcal{K}_w = \{K \in \mathcal{K} : w(K) > 0\}$. $\Diamond$

Notice that equation (2.2) may be written in matricial form as

$$M(\mathcal{K})w = m\,\mathbf{1},$$

and balanced weightings and fractional (including integer) elements of $D(\mathcal{K})$ are in one to one correspondence. Recall that $D(\mathcal{K})$ is always bounded, and we defined a balanced family as a family $\mathcal{K}$ for which $D(\mathcal{K}) \neq \emptyset$. Since it is defined by rational inequalities, the extreme points of $D(\mathcal{K})$ (when not empty) are always rational, and it follows that:

Lemma 2.3. *$\mathcal{K}$ is balanced if and only if there exists a balanced weighting of $\mathcal{K}$.*

Definition 2.4. A *partition* is a balanced weighting of multiplicity 1 (in this case the support is a set partition of N), and it is a *sum of partitions* if there exist partitions $w^j : \mathcal{K} \to \{0, 1\}$ and non-negative integers α_j, $j = 1, \dots, \ell$, such that $w = \sum_{j=1}^{\ell} \alpha_j w^j$. ◊

Definition 2.5. A family of coalitions $\mathcal{K}$ is called *partitionable* if any balanced weighting of $\mathcal{K}$ is a sum of partitions. ◊

Since we are assuming $\emptyset \notin \mathcal{K}$, and therefore $M(\mathcal{K})$ has no zero columns and $D(\mathcal{K})$ is bounded, it follows that:

Proposition 2.6. *$\mathcal{K}$ is partitionable if and only if $D(\mathcal{K})$ has no fractional extreme points.*

In particular, $\mathcal{K}$ is partitionable if $D(\mathcal{K}) = \emptyset$, i.e., whenever $\mathcal{K}$ is unbalanced.

3 Stability

In this section we introduce a concept of stability which is a variant of others considered before (see the Introduction). We will show results similar to theorem 1.8, e.g., that stable families and partitionable families coincide.

Associated with any family of coalitions $\mathcal{K}$, Boros, Gurvich and Vasin [4] considered a particular superadditive characteristic function which we will use extensively. Denoting by $\mathcal{K}'$ the family obtained from $\mathcal{K}$ by adding all the unions of pairwise disjoint coalitions in $\mathcal{K}$, they defined $v_{\mathcal{K}}$ by

$$v_{\mathcal{K}}(K) = \max\{|K'| : K' \in \mathcal{K}', K' \subset K\},$$

and proved:

Lemma 3.1. *The characteristic function $v_{\mathcal{K}}$ is superadditive, takes nonnegative integer values and satisfies*

$$v_{\mathcal{K}}(K) = |K| \textit{ for all } K \in \mathcal{K}' \quad \textit{and} \quad v_{\mathcal{K}}(K) < |K| \textit{ for all } K \notin \mathcal{K}'.$$

Notice that if $\mathcal{K}$ is a family of coalitions and w is a balanced weighting of $\mathcal{K}$, then $\mathcal{K}_w \subset \mathcal{K}$. For simplicity of notation, in this case we write v_w for $v_{\mathcal{K}_w}$.

Definition 3.2. We will say that a family of coalitions $\mathcal{K}$ is *stable* if either it is unbalanced or $C(v_w, \mathcal{K}) \neq \emptyset$ for every balanced weighting w of $\mathcal{K}$. ◊

The discussion of the primal/dual pair of linear programs in (1.3) and (1.5), and the fundamental relationship (1.4) lead immediately to:

Lemma 3.3. *A balanced family $\mathcal{K}$ is stable if and only if*

$$z^*(v_w, \mathcal{K}) = \max\left\{v_w(\mathcal{K}) \cdot y : y \in D(\mathcal{K})\right\} \leq v_w(N) \tag{3.4}$$

for every balanced weighting w of $\mathcal{K}$.

We turn now to the study of the relations between stability and partitionability.

Theorem 3.5. *If $\mathcal{K}$ is stable then it is partitionable.*

Proof. By proposition 2.6, if $D(\mathcal{K}) = \emptyset$ then $\mathcal{K}$ is partitionable, so let us assume $D(\mathcal{K}) \neq \emptyset$, and let y be a fractional extreme point of $D(\mathcal{K})$.

Fix $m \in \mathbb{Z}$, $m > 1$, such that $w = my$ is integral. Then, $M(\mathcal{K})w = m\,\mathbf{1}$, and w is a balanced weighting of $\mathcal{K}$. By lemma 3.3, the inequality (3.4) holds and therefore

$$\begin{aligned} v_w(N) &\geq \sum_{K \in \mathcal{K}} v_w(K) y_K = \sum_{K \in \mathcal{K}} |K_j| y_K \\ &= \sum_{K \in \mathcal{K}} \sum_{i \in N} \chi(K, i) y_K = \sum_{i \in N} \sum_{K \in \mathcal{K}} \chi(K, i) y_K. \end{aligned} \tag{3.6}$$

Now, since $y \in D(\mathcal{K})$,

$$\sum_{K \in \mathcal{K}} \chi(K, i) y_K = (M(\mathcal{K})y)_i = 1 \quad \text{for all } i \in N,$$

and (3.6) implies $v_w(N) \geq |N|$ $(= n)$. Moreover, by definition of v_w, we must have $v_w(N) \leq |N|$, and therefore equality holds in (3.6), i.e.,

$$|N| = \max_{K : K \in \mathcal{K}'_w} |K|.$$

In turn, lemma 3.1 implies that $N \in \mathcal{K}'_w$, and either $N \in \mathcal{K}_w$ or it is a disjoint union of elements of $\mathcal{K}_w$.

Let $\widehat{\mathcal{K}}$ be a family of disjoint elements of $\mathcal{K}$ (eventually with just one member) such that $\cup_{K \in \widehat{\mathcal{K}}} K = N$, and define $w^1 \in \{0, 1\}^{\mathcal{K}}$ by

$$w^1_K = \begin{cases} 1 & \text{if } K \in \widehat{\mathcal{K}}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $M(\mathcal{K})w^1 = \mathbf{1}$, then $w^1 \in D(\mathcal{K})$. Now, if $w^2 = w - w^1$, we obtain

$$\begin{aligned} M(\mathcal{K})w^2 &= M(\mathcal{K})w - M(\mathcal{K})w^1 = (m-1)\,\mathbf{1}, && \text{since } M(\mathcal{K})w = m\,\mathbf{1}, \\ w^2_K &= m y_K - w^1_K \geq 0, && \text{since } my \text{ is integral,} \end{aligned}$$

so that $\frac{1}{m-1} w^2 \in D(\mathcal{K})$. Moreover,

$$y = \frac{1}{m} w = \frac{1}{m} w^1 + \frac{m-1}{m} \left(\frac{1}{m-1} w^2 \right).$$

Since y is an extreme point of $D(\mathcal{K})$, we must have $y = w^1$, but then y is not fractional. □

If w is a balanced weighting of $\mathcal{K}$ (and $\mathcal{K}$ is balanced), the definitions of $\mathcal{K}_w$ and v_w imply that if $\widehat{\mathcal{K}}$ is a disjoint family of elements of $\mathcal{K}$, then

$$\sum_{K \in \widehat{\mathcal{K}}} v_w(K) \leq v_w(N). \tag{3.7}$$

Indeed, these inequalities are implied by superadditivity (inequalities (1.1)), and the characteristic functions v_w are superadditive (lemma 3.1). However, the inequalities (3.7) are weaker, and motivate the following definition:

Definition 3.8. The characteristic function v is a $\mathcal{W}$*-function (for the family* $\mathcal{K}$*)* if

$$\sum_{K \in \widehat{\mathcal{K}}} v(K) \leq v(N) \quad \text{for all set partitions } \widehat{\mathcal{K}} \subset \mathcal{K}. \tag{3.9}$$

(If there is no set partition contained in $\mathcal{K}$, then any characteristic function is a $\mathcal{W}$-function.) ♢

We show now that partitionability implies a seemingly stronger condition than stability:

Theorem 3.10. *If* $\mathcal{K}$ *is partitionable then* $C(v, \mathcal{K}) \neq \emptyset$ *for every* $\mathcal{W}$*-function* v.

Proof. If $\mathcal{K}$ is unbalanced, the result follows from lemma 1.6.

Suppose now $D(\mathcal{K}) \neq \emptyset$. Since $\mathcal{K}$ is partitionable, using proposition 2.6 we see that for any given characteristic function v,

$$\max \{v(\mathcal{K}) \cdot y : y \in D(\mathcal{K})\}$$

is attained at a 0-1 extreme point $\widehat{y}$ of $D(\mathcal{K})$. Then $\widehat{y}$ is a balanced weighting and its support $\widehat{\mathcal{K}}$ is a set partition of N, $\widehat{\mathcal{K}} \subset \mathcal{K}$. Hence

$$\sum_{K \in \mathcal{K}} v(K)\, y_K = \sum_{K \in \widehat{\mathcal{K}}} v(K).$$

If v is a $\mathcal{W}$-function, the right hand side in this equality is bounded above by $v(N)$, since the inequality (3.9) holds, and therefore $\widehat{y} \in C(v, \mathcal{K}) \neq \emptyset$. ☐

Using the previous results we obtain the equivalence between stability and partitionability, analogous to theorem 1.8:

Theorem 3.11. *A family* $\mathcal{K}$ *is stable if and only if it is partitionable.*

Proof. If the family is unbalanced it is stable (by definition) and partitionable (proposition 2.6). On the other hand, if $\mathcal{K}$ is balanced and stable, by theorem 3.5 it is partitionable. Finally, if $\mathcal{K}$ is balanced and partitionable, the stability follows from theorem 3.10 and lemma 3.3. ☐

4 Adding a single player to obtain stability

We now address the question on how can players be added to the coalitions of an unstable family $\mathcal{K}$ in order to obtain a stable family $\widetilde{\mathcal{K}}$, thus allowing the modification of the original coalitions, but not the addition of newer coalitions. In particular, we will have $|\mathcal{K}| = |\widetilde{\mathcal{K}}|$.

In this section we consider the addition of just one player, $n+1$, to the set of players $N = \{1, 2, \dots, n\}$, obtaining the new set of players $\widetilde{N} = N \cup \{n+1\}$. We will assume that $\mathcal{K}$ is unstable, and that the player $n+1$ has been included in a subset S of coalitions in $\mathcal{K}$. Thus, for $\widetilde{K} \in \widetilde{\mathcal{K}}$ either $\widetilde{K} = K \cup \{n+1\}$ for some $K \in S$, or $\widetilde{K} = K$ for some $K \in \mathcal{K} \setminus S$.

For simplicity of notation, throughout this section we let $A = M(\mathcal{K})$, and denote its rows by a_i, $i = 1, \dots, n$, so that for $K \in \mathcal{K}$,

$$a_{i,K} = \chi(K, i) = \begin{cases} 1 & \text{if } i \in K, \\ 0 & \text{otherwise.} \end{cases}$$

We keep the indices $\mathcal{K}$ for the columns of $\widetilde{A} = M(\widetilde{K})$, and assume that $\widetilde{A}$ has been obtained from $A = M(\mathcal{K})$ by appending the $(n+1)$-th row a_{n+1}, where

$$a_{n+1,K} = \begin{cases} 1 & \text{if } K \in S, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{K}$ is unstable, then it is balanced and not partitionable, i.e., $D(\mathcal{K}) = \big\{y \in \mathbb{R}^{\mathcal{K}} : Ay = \mathbf{1}, y \geq \mathbf{0}\big\}$ is bounded, non empty, and has fractional (in fact, non integral) extreme points. The polyhedron $D(\widetilde{\mathcal{K}})$ is obtained by adding the equality $a_{n+1} \cdot y = 1$ to the linear system defining $D(\mathcal{K})$, so that $D(\widetilde{\mathcal{K}}) \subset D(\mathcal{K})$.

Hence, if we want $\widetilde{\mathcal{K}}$ to be stable we have to eliminate all fractional extreme points of $D(\mathcal{K})$, either by leaving only integer extreme points or by making $\widetilde{\mathcal{K}}$ unbalanced. In general it will be impossible to eliminate all fractional extreme points by using just one equality. Even if possible, the task might be too difficult, although for some particular cases it might be easy (see, e.g., theorem 4.9).

It is much simpler to add one equality, via the 0-1 vector a_{n+1}, which is incompatible with the ones defining $D(\mathcal{K})$, so as to make $D(\widetilde{\mathcal{K}}) = \emptyset$, and this is the route we will follow.

Notice that by lemma 1.6, if $\widetilde{\mathcal{K}}$ is unbalanced, the definition of stability in terms of balanced weightings of $\widetilde{\mathcal{K}}$ is immaterial, and we may consider any characteristic function on $\widetilde{N}$.

There is also the question on how to relate characteristic functions originally defined on N to characteristic functions defined on $\widetilde{N}$, and vice-versa. Since we are only interested in the values the characteristic function takes on $\mathcal{K}$ (or $\widetilde{\mathcal{K}}$), and transforming $\mathcal{W}$-functions into $\widetilde{\mathcal{W}}$-functions is not an issue, a straightforward option is to extend or restrict them according to the definition of S, i.e., $v(K) = \widetilde{v}(\widetilde{K})$ (assuming $K = \widetilde{K}$ or $K \cup \{n+1\} = \widetilde{K}$). As is easily seen, the choice is quite arbitrary and it might depend on the concrete situation to be modeled. For instance, it could be that the addition of the player modifies the worths of the coalitions (even if the latter remain unmodified).

From another point of view, if $\widetilde{v}$ is a characteristic function defined on $\widetilde{N}$, and $x \in C(\widetilde{v}, \widetilde{\mathcal{K}})$, the possibility arises that $x_{n+1} = 0$, i.e., that the added player neither gets nor receives, and this may or may not be a desirable situation.

Let us turn to the study of the polyhedral structure of the $\widetilde{\mathcal{K}}$-cores, when the extended family of coalitions is unbalanced.

Lemma 4.1. *If $\mathcal{K}$ is unstable, $\widetilde{\mathcal{K}}$ is unbalanced, and $\widetilde{v}$ is a characteristic function on $\widetilde{N}$, then:*

1. *$C(\widetilde{v},\widetilde{\mathcal{K}})$ is not empty and unbounded.*
2. *The rays of $C(\widetilde{v},\widetilde{\mathcal{K}})$ (the directions in which $C(\widetilde{v},\widetilde{\mathcal{K}})$ is unbounded) are independent of $\widetilde{v}$.*
3. *If $(d,d_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ is a ray of $C(\widetilde{v},\widetilde{\mathcal{K}})$, then $d_{n+1} \neq 0$.*

Proof. The first part follows from lemma 1.6.

Writing $C(\widetilde{v},\widetilde{\mathcal{K}})$ in the form

$$\left\{(x,x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : A^T x + x_{n+1} a_{n+1} \geq \widetilde{v}(\widetilde{\mathcal{K}}), \mathbf{1}\cdot x + x_{n+1} \leq \widetilde{v}(\widetilde{\mathcal{K}})\right\},$$

we see that its rays, written in the form $(d,d_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$, are characterized as the non trivial solutions of the system of inequalities

$$\begin{array}{rl} A^T d \,+\, d_{n+1} a_{n+1} & \geq \mathbf{0}, \\ \mathbf{1}\cdot d \,+\, d_{n+1} & \leq 0, \end{array} \tag{4.2}$$

which is independent of $\widetilde{v}$.

Since, by hypothesis, the polyhedron $D(\mathcal{K})$ is not empty, for any characteristic function v defined on N the program min $\left\{\mathbf{1}\cdot x : A^T x \geq v(\mathcal{K})\right\}$ is feasible and bounded. Hence, the rays (d,d_{n+1}) of $C(\widetilde{v},\widetilde{\mathcal{K}})$ cannot have $d_{n+1}=0$. □

Thus, provided $\mathcal{K}$ is unstable and $\widetilde{\mathcal{K}}$ is unbalanced, the rays of a $\widetilde{\mathcal{K}}$-core (no matter what the corresponding characteristic function is), are positive multiples of rays of the form $(d,\pm 1)$, and their existence depends only on the matrix $\widetilde{A}$.

If there is a ray of the form $(d,1)$, there are allocations in which the added player, $n+1$, receives as much as desired. Similarly, if there exists a ray of the form $(d,-1)$, there are allocations in which the added player must pay (receive a negative quantity) as much as desired. Of course, in either case individual rationality will be violated, i.e., for any $\widetilde{v}$ defined on $\widetilde{N}$ there will exist $i \in \widetilde{N}$ and $x \in C(\widetilde{v},\widetilde{\mathcal{K}})$ for which $x_i < \widetilde{v}(\{i\})$.

For a ray of the form $(d,1)$, the system of inequalities (4.2) reduces to

$$\begin{aligned} A^T d &\geq -a_{n+1}, \\ \mathbf{1}\cdot d &\leq -1, \end{aligned}$$

which is equivalent to saying that the program

$$\min \left\{\mathbf{1}\cdot d : A^T d \geq -a_{n+1}\right\}, \tag{4.3}$$

is feasible and has optimal value $\xi_1^* \leq -1$. We observe also that the dual of this program is

$$\max \left\{-a_{n+1}\cdot y : Ay = \mathbf{1}, y \geq \mathbf{0}\right\} = \max \left\{-a_{n+1}\cdot y : y \in D(\mathcal{K})\right\}. \tag{4.4}$$

Similarly, for a ray of the form $(d,-1)$, we obtain the program

$$\min \left\{\mathbf{1}\cdot d : A^T d \geq a_{n+1}\right\}, \tag{4.5}$$

with optimal value $\xi^*_{-1} \leq 1$, and its dual,

$$\max \left\{ a_{n+1} \cdot y : y \in D(\mathcal{K}) \right\}. \tag{4.6}$$

We notice that since we are assuming $D(\mathcal{K}) \neq \emptyset$, its boundedness implies that all four programs (4.3)–(4.6) are always feasible and bounded.

There are a number of ways—perhaps too many—by which we can add the player $n+1$ to some of the coalitions so as to make $\widetilde{\mathcal{K}}$ unbalanced, and we will mention just a few simple ones.

4.1 Dummy player

A first possibility is to consider that the player $n+1$ is a *dummy* or *excluded* player (with respect to the family $\widetilde{\mathcal{K}}$), that is, she does not participate in any coalition in $\widetilde{\mathcal{K}}$. With our previous notations, we have $S = \emptyset$ and $a_{n+1} = \mathbf{0}$. Certainly the added equation, $a_{n+1} \cdot y = 0$, makes the system $\widetilde{A} y = \mathbf{1}$ infeasible, and therefore $D(\widetilde{\mathcal{K}}) = \emptyset$, and $\widetilde{\mathcal{K}}$ is unbalanced, stable and partitionable.

Lemma 4.7. *If $S = \emptyset$, then for any characteristic function $\widetilde{v}$ defined on $\widetilde{N}$, $C(\widetilde{v}, \widetilde{\mathcal{K}})$ has no ray of of the form $(d, 1)$. Furthermore, $(\mathbf{0}, -1)$ is a ray of $C(\widetilde{v}, \widetilde{\mathcal{K}})$.*

Proof. Since $a_{n+1} = \mathbf{0}$, $(\mathbf{0}, -1)$ is a non trivial solution of the system (4.2), and therefore is a ray. In addition, the programs (4.4) and (4.6) coincide and take the form

$$\max \left\{ 0 : y \in D(\mathcal{K}) \right\},$$

and therefore $\xi^*_1 = \xi^*_{-1} = 0$, so that there is no ray of the form $(d, 1)$. □

Thus, the incorporation of a dummy player makes the family stable, but intuitively this is at the expense of having her to pay for at least some of what is missing. Even worst, she could always choose to pay for all of what is missing!

4.2 Mediator

A symmetrical possibility to the dummy player is to consider that, conversely, the player $n+1$ participates in every coalition in $\widetilde{\mathcal{K}}$.

In this case we may say that the added player is a *mediator*, since her incorporation to the set of players has transformed an unstable family of coalitions into a stable one.

With the notations used at the beginning of this section, in this case we have $S = \mathcal{K}$, and $a_{n+1} = \mathbf{1}$. Unlike the dummy player case, however, it is not immediately clear that $D(\widetilde{\mathcal{K}}) = \emptyset$.

Let us start with a simple observation:

Remark 4.8. If the grand coalition, N, is a member of a family of coalitions $\widehat{\mathcal{K}}$, then $\widehat{\mathcal{K}}$ is balanced.

For, suppose $N \in \widehat{\mathcal{K}}$ and let $\mathbf{e}_N$ be the corresponding vector of the canonical basis, i.e., the 0-1 vector having exactly one 1 in the N-th position. Then $M(\widehat{\mathcal{K}}) \mathbf{e}_N = \mathbf{1}$, so that $D(\widehat{\mathcal{K}}) \neq \emptyset$. ◊

In this sense, something more can be said:

Theorem 4.9. *If $D(\mathcal{K}) \neq \emptyset$ and $S = \mathcal{K}$, then $\widetilde{\mathcal{K}}$ is balanced if and only if $N \in \mathcal{K}$.*

Proof. If $\widetilde{\mathcal{K}}$ is balanced then $D(\widetilde{\mathcal{K}}) \neq \emptyset$. Recalling that a_i is the i-th row of A, if $y \in D(\widetilde{\mathcal{K}})$ then

$$y_K \geq 0 \text{ for all } K \in \mathcal{K} \qquad \text{and} \qquad a_i \cdot y = 1 \text{ for all } i = 1, 2, \ldots, n+1.$$

For any $K \in \mathcal{K}$ and any $i \in N$, we cannot have $y_K > 0$ and $i \notin K$, since in this case we would have $a_{i,K} = \chi(K, i) = 0$ and

$$1 = a_i \cdot y < \mathbf{1} \cdot y = 1.$$

Therefore, the only coalition in the support of y is N, i.e., $N \in \mathcal{K}$ and $y = \mathbf{e}_N$.

To see the converse, since $S = \mathcal{K}$ and $N \in \mathcal{K}$, then $\widetilde{N} \in \widetilde{\mathcal{K}}$. Remark 4.8 implies then that $\mathbf{e}_N \in D(\widetilde{\mathcal{K}}) \neq \emptyset$. ☐

Remark 4.10. Under the hypothesis of the previous result,

- $D(\widetilde{\mathcal{K}}) = \{\mathbf{e}_N\}$ and $\widetilde{\mathcal{K}}$ is partitionable (and stable).
- For any $\widetilde{\mathcal{W}}$-function $\widetilde{v}$ defined on $\widetilde{N}$, we have $z^*(\widetilde{v}, \widetilde{\mathcal{K}}) = \widetilde{v}(\widetilde{N})$, $C(\widetilde{v}, \widetilde{\mathcal{K}}) \neq \emptyset$, and $x(\widetilde{N}) = \widetilde{v}(\widetilde{N})$ for all $x \in C(\widetilde{v}, \widetilde{\mathcal{K}})$ (i.e., x is efficient). ◊

Let us turn now to the study of the rays when $S = \mathcal{K}$ and $\widetilde{\mathcal{K}}$ is unbalanced.

Lemma 4.11. *If $S = \mathcal{K}$ and $\widetilde{\mathcal{K}}$ is unbalanced, then for any characteristic function $\widetilde{v}$ on $\widetilde{N}$, $C(\widetilde{v}, \widetilde{\mathcal{K}})$ has no rays of the form $(d, -1)$.*

Consequently, $C(\widetilde{v}, \widetilde{\mathcal{K}})$ has rays of the form $(d, 1)$.

Proof. Since $a_{n+1} = \mathbf{1}$, the existence of rays of the form $(d, -1)$ is related to the linear program (4.5) and its dual (4.6), which we may write as

$$\xi^*_{-1} = \min \left\{ \mathbf{1} \cdot d : A^T d \geq \mathbf{1} \right\} = \max \left\{ \mathbf{1} \cdot y : y \in D(\mathcal{K}) \right\}.$$

Moreover, given that A is a 0-1 matrix, for the dual program we have

$$1 = a_i \cdot y \leq \mathbf{1} \cdot y \quad \text{for } i = 1, \ldots, n,$$

since $y \geq \mathbf{0}$ for $y \in D(\mathcal{K})$, and therefore $\xi^*_{-1} > 1$ unless for all $y \in D(\mathcal{K})$, all K in the support of y, and all $i \in N$ we have $i \in K$. But this would mean that N is in the support of y, and therefore, by remark 4.8, $\widetilde{\mathcal{K}}$ is balanced, contrary to the hypothesis.

We conclude that $\xi^*_{-1} > 1$ and therefore there are no rays of the form $(d, -1)$. ☐

Let us sum up our findings for the case of the mediator, assuming $\mathcal{K}$ is unstable.

If $N \notin \mathcal{K}$, the new family $\widetilde{\mathcal{K}}$ is stable but unbalanced. For any characteristic function $\widetilde{v}$ defined on $\widetilde{N}$, there will be allocations in the corresponding $\widetilde{\mathcal{K}}$-core which will assign the mediator a positive amount of money, as large as desired.

On the other hand, if $N \in \mathcal{K}$, we are in an ideal situation: $C(\widetilde{v}, \widetilde{N}) \neq \emptyset$ for any $\widetilde{\mathcal{W}}$-function $\widetilde{v}$ on $\widetilde{N}$, and any allocation in the $\widetilde{\mathcal{K}}$-core is efficient.

4.3 Adding a player to just one coalition

As a final example, let us consider the case where the player $n+1$ is added to just one coalition, K_0, of $\mathcal{K}$. That is, we have $S = \{K_0\}$ and $a_{n+1} = \mathbf{e}_{K_0}$, i.e.,

$$a_{n+1,K} = \begin{cases} 1 & \text{if } K = K_0, \\ 0 & \text{otherwise.} \end{cases}$$

As in the previous cases, we wonder whether this choice makes $D(\widetilde{\mathcal{K}})$ empty or not. Although it is easy to exhibit examples where an appropriate choice of K_0 will make $D(\widetilde{\mathcal{K}}) = \emptyset$, the following example shows that sometimes this choice cannot be made.

Example 4.12. If $\mathcal{K} = [N] \cup \mathcal{K}_0$ is unstable, then the addition of the player to a single arbitrary coalition makes $D(\widetilde{\mathcal{K}}) \neq \emptyset$, i.e., $\widetilde{\mathcal{K}}$ is balanced.

For, if the player is added to one of the individual coalitions, then the vector $y \in \mathbb{R}^{\mathcal{K}}$ defined by

$$y_K = \begin{cases} 1 & \text{if } K = \{i\} \text{ for some } i \in N, \\ 0 & \text{in any other case,} \end{cases}$$

belongs to $D(\widetilde{\mathcal{K}})$.

Similarly, if the player is added to a single coalition $K_j \in \mathcal{K}_0$, let $y \in \mathbb{R}^{\mathcal{K}}$ be defined by

$$y_K = \begin{cases} 1 & \text{if } K = K_j, \\ 1 & \text{if } K = \{i\} \text{ for some } i \in N \setminus K_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $y \in D(\widetilde{\mathcal{K}})$. ◊

Lemma 4.13. *If $K_0 \in \mathcal{K}$ and $S = \{K_0\}$ is such that $\widetilde{\mathcal{K}}$ is unbalanced, then for any characteristic function $\widetilde{v}$ on $\widetilde{N}$, there is no ray of $C(\widetilde{v}, \widetilde{\mathcal{K}})$ of the form $(d, 1)$, and consequently there are rays of the form $(d, -1)$.*

Moreover, $y_{K_0} < 1$ for every $y \in D(\mathcal{K})$.

Proof. The program (4.4) takes the form

$$\xi_1^* = \max \left\{ -y_{K_0} : y \in D(\mathcal{K}) \right\}.$$

Since A is a 0-1 matrix with no column of zeroes, this implies that $y_{K_0} \leq 1$ for all $y \in D(\mathcal{K})$, and hence $\xi_1^* \geq -1$. However, if $\xi_1^* = -1$, there is a solution $y \in D(\mathcal{K})$ with $y_{K_0} = 1$, and consequently $y \in D(\widetilde{\mathcal{K}})$, which is impossible since we are assuming $D(\widetilde{\mathcal{K}}) = \emptyset$. □

As was to be suspected, adding the player $n+1$ to just one coalition is an intermediate case between the dummy player and the mediator cases. It is similar to the dummy player in that, provided $\widetilde{\mathcal{K}}$ is unbalanced, for any characteristic function defined on $\widetilde{N}$ we may find allocations in the corresponding $\widetilde{\mathcal{K}}$-core for which she pays as much as desired. Unlike the dummy player and resembling the mediator, sometimes the resulting family $\widetilde{\mathcal{K}}$ may be balanced, as example 4.15 shows.

4.4 Examples

Example 4.14. Let $\mathcal{K} = \{\{2,3\},\{2,4\},\{3,4\},\{1,5\},\{1\}\}$. In this case,

$$M(\mathcal{K}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad D(\mathcal{K}) = \{(1/2,1/2,1/2,1,0)\},$$

and the family is balanced but unstable.

If we add one player to every coalition, a mediator, there is no solution in the core for which this new player looses as much as we want. Nevertheless, there is a solution in which she obtains some profit and we can find it by complementarity conditions. For example, $(-1,-1/2,-1/2,-1/2,0,1)$ is a ray of $C(\widetilde{v},\widetilde{\mathcal{K}})$ for any characteristic function $\widetilde{v}$ on $\widetilde{N}$.

It is clear that we can add one player to one coalition and the resulting family is balanced but unstable. In this case, it is enough to add a player to coalition number 4 and then $D(\widetilde{\mathcal{K}}) = D(\mathcal{K})$.

If we want to add the new player to only one coalition, in order to obtain an unbounded core we have to add it to any of the three first coalitions. Adding her to the first coalition we have that $(1,-1/2,0,-1/2,-1,-1)$ is a ray of $C(\widetilde{v},\widetilde{\mathcal{K}})$ for any characteristic function $\widetilde{v}$ on $\widetilde{N}$. ◊

Example 4.15. Let $N = \{1,2,3\}$ and $\mathcal{K} = [N] \cup \mathcal{K}_0 = [N] \cup \{\{1,2\},\{1,3\},\{2,3\}\}$. Then,

$$M(\mathcal{K}) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix},$$

$$D(\mathcal{K}) = \{(0,0,0,1/2,1/2,1/2),(0,0,1,1,0,0), (0,1,0,0,1,0),(1,0,0,0,0,1),(1,1,1,0,0,0)\},$$

and the family is balanced but unstable.

In this case, adding one player to only one coalition in $\mathcal{K}_0$ always makes the family balanced and stable. ◊

5 Adding several players to obtain stability

In the previous section we have seen that we can bring stability to an unstable family by adding just one player. However, since we asked for unbalancedness of the new family, this was at the expense that for any characteristic function there will always be allocations in the corresponding $\widetilde{\mathcal{K}}$-core which violate individual rationality. One remedy, of course, is to explicitly ask for individual rationality.

Thus, in this section we consider the effect of adding players to an unstable family of the form $\mathcal{K} = \mathcal{K}_0 \cup [N]$, with $\mathcal{K}_0 \cap [N] = \emptyset$. Unlike the preceding section, when extending the family we now include the individual coalitions formed by the added players. Therefore, if the new set of players is $\widetilde{N}$, the new family of coalitions is of the form $\widetilde{\mathcal{K}} = \widetilde{\mathcal{K}}_0 \cup [\widetilde{N}]$, and the new players are added to (some of) the coalitions in $\mathcal{K}_0$, so that $|\widetilde{\mathcal{K}}_0| = |\mathcal{K}_0|$.

As discussed in the Introduction, the polyhedra $D(\mathcal{K})$ and $D(\widetilde{\mathcal{K}})$ are now always non empty, so that both $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are balanced. By the results in section 3, stability is equivalent to partitionability, and we are in the conditions of theorem 1.9.

Given a 0-1 matrix M, $G(M)$ denotes the graph with node set the set of columns of M and where two nodes are adjacent if there is a row in M with two ones in their corresponding positions.

Referring the reader to the Appendix for definitions and results from graph and combinatorial polyhedral theory, we see that we have only two possibilities for an unstable family $\mathcal{K}$: either the corresponding graph $G(M(\mathcal{K}_0))$ is not perfect, or this graph is perfect but $M(\mathcal{K}_0)$ is not a clique-node matrix.

If the graph $G(M(\mathcal{K}_0))$ is not perfect, we may bring it to perfection by adding edges (and hence players) so that there are no induced odd cycles or induced complements of odd cycles.

In all, the study of stability in the present context is reduced to the study of the polyhedron $P(\mathcal{K}_0) = \{x \geq 0 : M(\mathcal{K}_0)x \leq \mathbf{1}\}$, and the convex hull of its integer feasible solutions, $P(\mathcal{K}_0)^*$, so that stability of $\mathcal{K}$ is equivalent to having $P(\mathcal{K}_0) = P(\mathcal{K}_0)^*$.

Let us note that the integer solutions of $P(\mathcal{K}_0)$ correspond precisely to set partitions of N formed only with elements of $\mathcal{K}$. The following theorem shows that we can add players while keeping the original set partitions. Its proof also tells us how many players we need to add and where to add them.

Theorem 5.1. *Let $\mathcal{K} = \mathcal{K}_0 \cup [N]$ be an unstable family of coalitions. There is a set of players we can add to a set of coalitions in $\mathcal{K}_0$ such that the new family $\widetilde{\mathcal{K}} = \widetilde{\mathcal{K}}_0 \cup [\widetilde{N}]$ is stable and $P(\mathcal{K}_0)^* = P(\widetilde{\mathcal{K}}_0)^*$ if and only if the graph $G(M(\mathcal{K}_0))$ is perfect.*

Proof. Let us write $M = M(\mathcal{K}_0)$, $P = P(\mathcal{K}_0)$, $\widetilde{M} = M(\widetilde{\mathcal{K}}_0)$, $\widetilde{P} = P(\widetilde{M})$, $P^* = P(\mathcal{K}_0)^*$, $\widetilde{P}^* = P(\widetilde{\mathcal{K}}_0)^*$,

($\Rightarrow$) Dominated rows in M do not affect the polyhedron P, therefore we may assume that we have a minimal description of P. In case we have dominated inequalities (players) we can delete them, analyze the family without them and add them back after obtaining a stable family $\widetilde{\mathcal{K}}$. We will come back to this kind of situation in remark 5.2.

P is a relaxation of P^*, and the addition of players corresponds to the addition of 0-1 rows to the matrix M. By hypothesis, by adding them we obtain the matrix $\widetilde{M}$ which defines P^*, and therefore this convex hull admits a description by 0-1 inequalities.

By antiblocker duality, the rows of $\widetilde{M}$ (after removal of dominated rows) are the non zero extreme points of its antiblocker polyhedron, $\widetilde{P}^c$. If the extreme points of this polyhedron are integral, it coincides with the convex hull of its integer solutions, that is, $\widetilde{P}^c = P(a(M))^c$ (see diagram (9.1) in the Appendix). Since these two polyhedra coincide, so do P^* and $P(a(a(M)))$. Moreover, since M is not perfect, the matrix M is not a clique-node matrix but $G(M) = G(\widetilde{M})$ and $G(M)$ is perfect.

($\Leftarrow$) Since the graph $G(M)$ is perfect and M is not a clique-node matrix, there is a set of cliques $Q_1, \ldots, Q_\ell$ that do not have its incidence vector within the rows of $G(M)$. Therefore, adding ℓ players, $i_1, \ldots, i_\ell$, in such a way that player i_k

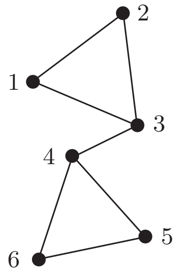

Figure 5.1: The graph $G(\mathcal{K}_0)$ in the example 5.3.

participates precisely in the coalitions determining Q_k, $k = 1, \ldots, \ell$, we obtain a clique-node matrix of the graph $G(M)$. Again, from Chvátal's result the new family obtained after this addition is stable. □

Remark 5.2. Consider a family including all the individual coalitions. Suppose that player 1 participates in a set A_1 of coalitions, player 2 in A_2 coalitions and that $A_1 \subsetneq A_2$. It is clear that player 1 does not change the stability or instability of the family and we call her *unessential player*.

In this case, in the problem (1.5) we can replace the variable y_2 by $y_1 + y_2$ and solve the problem without y_1, which can be performed taking superaditivity of the characteristic function into account. This corresponds to the deletion of the unessential player in the original family of coalitions and allows working in lower dimensions. ◊

Example 5.3. Let $N = \{1, \ldots, 7\}$ and

$$\mathcal{K} = \{\{1,3\},\{1,2\},\{2,3,7\},\{4,6,7\},\{5,6\},\{4,5\}\} \cup [N].$$

In this case,

$$M(\mathcal{K}_0) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

It is clear that the family $\mathcal{K}$ is unstable. Moreover, $M(\mathcal{K}_0)$ is not the clique-node matrix of $G(\mathcal{K}_0)$ (Figure 5.1), which is a perfect graph.

The set of maximal cliques that are not rows of $M(\mathcal{K}_0)$ is

$$F = \{\{1,2,3\},\{4,5,6\}\}.$$

From the previous lemma we know that if we add two players (one to the first three coalitions and another to the last three coalitions) we obtain a stable family that preserves the set of partitions of the given family.

That is,

$$\widetilde{\mathcal{K}}_0 = \{\{1,3,8\},\{1,2,8\},\{2,3,7,8\},\{4,6,7,9\},\{5,6,9\},\{4,5,9\}\},$$

$M(\widetilde{\mathcal{K}}_0)$ is a perfect matrix, and $\widetilde{\mathcal{K}} = \widetilde{\mathcal{K}}_0 \cup [\{1, \ldots, 9\}]$ is stable. ◊

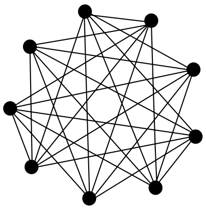

Figure 5.2: Moon-Moser graph G_3.

Let us present a discouraging result: the number of players we need to add in order to obtain a stable family can be exponentially large, even under the hypothesis of theorem 5.1.

Example 5.4. A Moon-Moser graph $G_n = (V_n, E_n)$ of complexity n contains n components $S_1, S_2, \ldots, S_n$ such that

- Every component S_i has 3 isolated nodes, $1 \leq i \leq n$.
- $E_n = \{(u, v) : u \in S_i,\, v \in S_j \text{ and } i \neq j\}$.

A Moon-Moser graph contains 3^n cliques. Clearly, every Moon-Moser graph is perfect since the complementary graph is a set of disjoint triangles. In figure 5.2 we present the Moon-Moser graph G_3. ♢

In order to avoid this bad behavior it could be reasonable, in the context of game theory, to limit the size of the coalitions. In this case, we also limit the number of coalitions we need to add in order to obtain stability.

6 Deletion of coalitions to obtain stability

Continuing the study of families of coalitions of the form $\mathcal{K} = \mathcal{K}_0 \cup [N]$ started in section 5, in this section we analyze the effect of deleting coalitions.

Since deletion of coalitions corresponds to deletion of nodes in the graph $G(M(\mathcal{K}_0))$, our problem is to find a set of nodes of minimum cardinality such that deleting them from $G(M(\mathcal{K}_0))$ we obtain a perfect graph. It is clear that we can always find such a set of coalitions.

We have the following:

Lemma 6.1. *Given an unstable family of coalitions* $\mathcal{K} = \mathcal{K}_0 \cup [N]$ *such that* $M(\mathcal{K}_0)$ *is an* $n \times m$ *clique-node matrix, there is a set of* k *coalitions,* $1 \leq k \leq m - 1$*, such that deleting them from* $\mathcal{K}_0$ *the resulting family* $\widetilde{\mathcal{K}}$ *is stable.*

Proof. Since $M(\mathcal{K}_0)$ is a clique-node matrix, the graph $G(\mathcal{K}_0)$ is imperfect. It is clear that there is a set of nodes we can delete from $G(\mathcal{K}_0)$ in order to destroy all the forbidden structures in the graph. Let us assume that the minimum number of nodes is k, and since nodes correspond to coalitions, let $\{C_1, \ldots, C_k\}$ be such a set of nodes in $G(M(\mathcal{K}_0))$. Therefore $G' = G(M(\mathcal{K}_0)) \setminus \{C_1, \ldots, C_k\}$ is perfect. If $\widetilde{\mathcal{K}} = \mathcal{K}_0 \setminus \{C_1, \ldots, C_k\}$ then it is stable. □

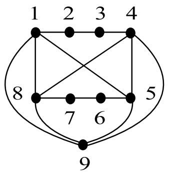

Figure 6.1: The graph $G(M(\mathcal{K}_0))$ in example 6.2.

Example 6.2. Let $\mathcal{K} = \mathcal{K}_0 \cup [\{1, \ldots, 10\}]$ so that

$$M(\mathcal{K}_0) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

and the graph $G(M(\mathcal{K}_0))$ is shown in figure 6.2.

In this case, it is easy to check that deleting only one coalition the family remains unstable (or equivalently, leaves the graph imperfect). Nevertheless, there is a set of two coalitions (for example, coalitions 1 and 5) such that after deleting them we obtain a perfect graph.

Let us suppose that we have deleted these two coalitions thus arriving to the family $\widetilde{\mathcal{K}}_0$ where

$$M(\widetilde{\mathcal{K}}_0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let us note that by deletion of coalitions we may arrive to unessential players, but in order to preserve a little more the structure of the original game we can preserve them in the game. ◊

If the graph $G(M(\mathcal{K}_0))$ is not perfect, we can delete coalitions in order to obtain a perfect subgraph of it, G', and then add players (without changing the graph G') in order to have a stable family.

Let us point out that deletion of coalitions and addition of players do not commute, that is, we may obtain different results depending on the order on which these operations are done. The following example illustrates this behavior.

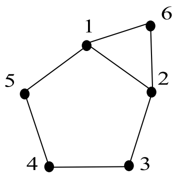

Figure 6.2: The graph $G(\mathcal{K}_0)$ in the example 6.3.

Example 6.3. Let $N = \{1, \dots, 6\}$ and

$$\mathcal{K} = \{\{1,5,7\},\{1,2,6\},\{2,3\},\{3,4\},\{4,5\},\{6,7\}\} \cup [N].$$

In this case,

$$M(\mathcal{K}_0) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and it is readily seen that the family $\mathcal{K}$ is unstable. Moreover, $M(\mathcal{K}_0)$ is not the clique-node matrix of $G(\mathcal{K}_0)$ (Figure 6.2) which is not a perfect graph.

There is only one clique whose incidence vector is not a row of $M(\mathcal{K}_0)$ is

$$F = \{\{1,2,6\}\}.$$

Then, adding one player to the first, the second and the 6-th coalitions, $M(\widetilde{\mathcal{K}})$ is a clique-node matrix (with unessential players). That is,

$$\widetilde{\mathcal{K}}_0 = \{\{1,5,7,8\},\{1,2,6,8\},\{6,7,8\},\{2,3\},\{3,4\},\{4,5\}\}.$$

Nevertheless, the graph $G(M(\widetilde{\mathcal{K}_0}))$ did not change, that is, it is not perfect. In order to make the family $\widetilde{\mathcal{K}}$ stable we can delete any of the first five new coalitions.

If we delete any coalition but the sixth, the resulting graph is perfect. Suppose we delete the first coalition. In this case, there is no need to add players since the submatrix of $M(\mathcal{K}_0)$ obtained after deleting the first column is a clique-node matrix. ♢

7 Instability index of a family

In polyhedral combinatorics and graph theory there are different kinds of indices defined in order to measure a certain property such as the disjunctive index of a matrix [1] or the imperfection index of a graph [11], among others.

In our case, it is appropriate to measure the instability of a family of coalitions.

Given an unstable family of coalitions $\mathcal{K} = \mathcal{K}_0 \cup [N]$ such that $M(\mathcal{K}_0)$ is an $n \times m$ matrix, we have considered two important numbers: $c(\mathcal{K})$ and $p(\mathcal{K})$.

$c(\mathcal{K})$ is the minimum number of coalitions we have to delete to obtain a perfect subgraph of $G(M(\mathcal{K}_0))$, and $p(\mathcal{K})$ is the minimum number of players we need to add to obtain a stable family when the associated graph is perfect. It is clear that $c(\mathcal{K})$ is bounded from above by the number of coalitions, and that $p(\mathcal{K}) = 0$ and $c(\mathcal{K}) = 0$ correspond to a stable family $\mathcal{K}$.

Example 7.1. If $\mathcal{K}$ is the family of coalitions in example 5.3, $p(\mathcal{K}) = 2$ and $c(\mathcal{K}) = 0$. For the family in example 6.2, $c(\mathcal{K}) = 2$. ◊

Example 7.2. Consider the family in example 6.3. In order to have a stable family we have to obtain a clique-node matrix of a perfect graph. Let $\widetilde{\mathcal{K}}_1$ the family obtained after the deletion of the first coalition. Then $p(\widetilde{\mathcal{K}}_1) = 0$ and $c(\widetilde{\mathcal{K}}_1) = 0$.

Let $\widetilde{K}_2$ the family obtained from $\mathcal{K}$ after the addition of a player as in the example 6.2, then $p(\widetilde{\mathcal{K}}_2) = 0$ and $c(\widetilde{\mathcal{K}}_2) = 1$. ◊

Let us remark that deciding whether a 0-1 matrix is a clique-node matrix and deciding whether a graph is perfect are tasks that can be done in polynomial time (see [9, theorem 3.9] and [6, 7]).

Remark 7.3. In order to obtain the instability indices of a family $\mathcal{K} = \mathcal{K}_0 \cup [N]$, we may proceed as follows.

If the matrix $M(\mathcal{K}_0)$ is a clique-node matrix and $G(M(\mathcal{K}_0))$ is a perfect graph, then $p(\mathcal{K}) = c(\mathcal{K}) = 0$ and the family is stable.

If $G(M(\mathcal{K}_0))$ is a perfect graph, the number of players we need to add to make the matrix clique-node is $p(\mathcal{K})$ and $c(\mathcal{K}) = 0$.

If $G(M(\mathcal{K}_0))$ is not a perfect graph, there is a set of nodes (coalitions) of minimum cardinality, $c(\mathcal{K})$, such that after deleting it from the graph we obtain a perfect graph. Let $\widetilde{\mathcal{K}}$ denote the resulting family. Then, there is a set of players we can add to $\widetilde{\mathcal{K}}$ such that the new family is stable but in general $p(\mathcal{K}) \neq p(\widetilde{\mathcal{K}})$. ◊

As a final remark on this topic, we observe that determining the instability indices is in general a difficult task. Although it is possible to decide in polynomial time whether a matrix is clique-node or a graph is perfect, determining a minimum set of rows to be added in order to make a matrix clique-node may demand a non-polynomial amount of time.

8 Fractional participation of players

Referring the reader to the Introduction, in this section we consider that a player does not fully participate in the coalition, say, because of lack of time for meetings or problems in communicating with the other members. Or alternatively, we may think that the player participates with some probability in the coalition.

A traditional coalition $K \subset N$, may be identified with the characteristic 0-1 vector $x = (\chi(K, 1), \chi(K, 2), \ldots, \chi(K, n))$, so that, e.g., $x_i = 1$ means that the player i is in the coalition, and this suggests the following definition:

Definition 8.1. A *fractional coalition*, or f-coalition for short, will be a vector of the form $F = (a_{1,F}, a_{2,F}, \ldots, a_{n,F})$, where $a_{i,F} \in \mathbb{Q} \cap [0, 1]^n$, meaning that the player i participates in the f-coalition the fractional amount $a_{i,F}$. ◊

If $K \subset N$, the f-coalition $(\chi(K, 1), \chi(K, 2), \ldots, \chi(K, n))$ will be denoted by K_f. In particular, the grand coalition is written as the f-coalition

$$N_f = (1, 1, \ldots, 1) = \mathbf{1}.$$

Given a finite family $\mathcal{F}$ of f-coalitions, we may consider the matrix $M(\mathcal{F})$, with entries $a_{i,F}$ with $i \in N$ and $F \in \mathcal{F}$, and characteristic functions $v : \mathcal{F} \to \mathbb{R}$, identified with vectors in $\mathbb{R}^{\mathcal{F}}$, so that the $\mathcal{F}$-core becomes (cf. equation (1.2))

$$C(v, \mathcal{F}) = \left\{x \in \mathbb{R}^n : v(N_f) \geq x(N) \text{ and } M(\mathcal{F})^T x \geq v(\mathcal{F})\right\}.$$

The linear programs (1.3) and (1.5) are similar,

$$z^*(v, \mathcal{F}) = \min\{\mathbf{1} \cdot x : M(\mathcal{F})^T x \geq v(\mathcal{F})\},$$

and if

$$D(\mathcal{F}) = \{y \geq \mathbf{0} : M(\mathcal{F}) y = \mathbf{1}\},$$

then

$$z^*(v, \mathcal{F}) = \begin{cases} \max\{v(\mathcal{F}) \cdot y : y \in D(\mathcal{F})\} & \text{if } D(\mathcal{F}) \neq \emptyset, \\ -\infty \quad \text{for any } v & \text{if } D(\mathcal{F}) = \emptyset. \end{cases}$$

Also, the relation (1.4) holds,

$$C(v, \mathcal{F}) \neq \emptyset \quad \text{if and only if} \quad z^*(v, \mathcal{F}) \leq v(N_f),$$

and we may define (cf. definition 1.7):

Definition 8.2. The finite family $\mathcal{F}$ of f-coalitions is *balanced* if $D(\mathcal{F}) \neq \emptyset$. ◊

Remark 8.3. In what follows, all the families of f-coalitions will be assumed to be finite. ◊

The definition of f-coalition forces us to extend notions such as cardinality or subset. It is natural to define the *f-cardinal* or *f-count* of the f-coalition $F = (a_{1,F}, a_{2,F}, \ldots, a_{n,F})$ by

$$\sigma(F) = \sum_{i \in N} a_{i,F},$$

so that for $K \subset N$ we have $\sigma(K_f) = |K|$.

Similarly, we may say that given two f-coalitions, F_1 and F_2, then F_1 is a *f-subcoalition* of F_2, in symbols $F_1 \sqsubset F_2$, if $a_{i,F_1} \leq a_{i,F_2}$ for all $i \in N$.

However, deciding whether two f-coalitions, F_1 and F_2, are disjoint is more delicate. It seems natural to define it by the condition $\max_i a_{i,F_1} \times a_{i,F_2} = 0$, but it will be more convenient to define it as follows:

Definition 8.4. The f-coalitions F_1 and F_2, $F_1 \neq F_2$, are *f-disjoint* if

$$\sum_{i \in N} (a_{i,F_1} + a_{i,F_2}) \leq 1. \qquad \Diamond$$

If F_1 and F_2 are f-disjoint, their *f-disjoint union* is defined as (cf. [5])

$$F_1 \sqcup F_2 = \left(a_{1,F_1} + a_{1,F_2}, a_{2,F_1} + a_{2,F_2}, \ldots, a_{n,F_1} + a_{n,F_2}\right),$$

so that

$$\sigma(F_1 \sqcup F_2) = \sigma(F_1) + \sigma(F_2).$$

Given a subfamily $\widehat{\mathcal{F}}$ of pairwise f-disjoint f-coalitions, we may define the f-disjoint union $\sqcup_{F \in \widehat{\mathcal{F}}} F$ by induction, and $\widehat{\mathcal{F}}$ is a *f-partition* if

$$\sqcup_{F \in \widehat{\mathcal{F}}} F = N_f.$$

Example 8.5. Consider $N = \{1, 2, 3\}$ and $\mathcal{F} = \{F_1, F_2, F_3\}$ where

$$F_1 = (1, 0, 1/2), \quad F_2 = (0, 1, 1/2), \quad F_3 = (1/3, 1/3, 1/3).$$

Then,

$$M(\mathcal{F}) = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 1/2 & 1/2 & 1/3 \end{bmatrix}.$$

Since $(1, 1, 0) \in D(\mathcal{F})$, $\mathcal{F}$ is balanced. Also, F_1 and F_2 are f-disjoint and $N_f = F_1 \sqcup F_2$, so that $\{F_1, F_2\}$ is a f-partition (but not a set partition). ♢

As was the case with the usual coalitions, f-partitions and 0-1 extreme points of $D(\mathcal{F})$ are in a one to one correspondence. For, given an f-partition $\widehat{\mathcal{F}}$ contained in $\mathcal{F}$, we may define $y \in \mathbb{R}^{\mathcal{F}}$ by

$$y_F = \begin{cases} 1 & \text{if } F \in \widehat{\mathcal{F}}, \\ 0 & \text{if } F \in \mathcal{F} \setminus \widehat{\mathcal{F}}, \end{cases} \tag{8.6}$$

and it is clear that $y \in D(\mathcal{F})$. Similarly, if $y \in D(\mathcal{F})$ is 0-1, its support,

$$\operatorname{supp}(y) = \{F \in \mathcal{F} : y_F > 0\},$$

is an f-partition.

In the present context, it is rather involved to consider balanced weightings as we did in section 2. Instead, it is simpler to work directly with elements of $D(\mathcal{F})$, keeping implicitly the one to one correspondence between balanced weightings and (fractional) points in $D(\mathcal{F})$.

Definition 8.7 (cf. proposition 2.6)**.** A family $\mathcal{F}$ of f-coalitions is *partitionable* if either $D(\mathcal{F}) = \emptyset$ or all the extreme points of $D(\mathcal{F})$ are 0-1. ♢

Example 8.8. Consider $N = \{1, 2, 3\}$ and $\mathcal{F} = \{F_1, F_2\}$ with

$$F_1 = (1, 0, 1/2), \quad F_2 = (0, 1, 1/3).$$

Then,

$$M(\mathcal{F}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1/2 & 1/3 \end{bmatrix},$$

and $D(\mathcal{F}) = \emptyset$. Thus, $\mathcal{F}$ is unbalanced and partitionable. ♢

Example 8.9. Consider $N = \{1, 2\}$ and $\mathcal{F} = \{F_1, F_2, F_3\}$ where

$$F_1 = (1, 0), \quad F_2 = (0, 1), \quad F_3 = (1/3, 1/2),$$

and therefore

$$M(\mathcal{F}) = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/2 \end{bmatrix}.$$

The extreme points of $D(\mathcal{F})$ are $(1, 1, 0)$ and $(1/3, 0, 2)$, so that $\mathcal{F}$ is balanced but not partitionable. ♢

Example 8.10. Consider N and $\mathcal{F}$ as in example 8.5. Although the extreme points of $D(\mathcal{F})$—$(1, 1, 0)$ and $(0, 0, 3)$—are integral, not all of them are 0-1, so that $\mathcal{F}$ is balanced and not partitionable. ♢

Example 8.11. Consider $N = \{1, 2, 3\}$ and $\mathcal{F} = \{F_1, F_2, F_3\}$ where

$$F_1 = (1, 0, 1/2), \quad F_2 = (0, 1, 1/2), \quad F_3 = (1, 1, 0).$$

Then,

$$M(\mathcal{F}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

Now, $D(\mathcal{F}) = \{(1, 1, 0)\}$ so that $\mathcal{F}$ is balanced and partitionable. ◊

8.1 Stability of fractional families

We turn now to extend the notion of stability to families of fractional coalitions.

To this end, given a family $\mathcal{F}$ of f-coalitions and $y \in D(\mathcal{F})$, let $\mathcal{F}_y$ be the family of all possible f-disjoint unions of elements of $\operatorname{supp}(y)$, and consider the (fractional) characteristic function v_y defined by

$$v_y(F) = \begin{cases} 0 & \text{if there is no } F' \in \operatorname{supp}(y) \text{ such that } F' \sqsubset F, \\ \max\{\sigma(F') : F' \in \mathcal{F}_y, F' \sqsubset F\} & \text{otherwise.} \end{cases}$$

Similar to lemma 3.1, for any f-coalition F we have:

$$0 \le v_y(F) \le \sigma(F),$$
$$v_y(F) = \sigma(F) \quad \text{if and only if } F \in \mathcal{F}_y,$$

and for $K \subset N$,

$$v_y(K_f) \le |N|,$$
$$v_y(K_f) = |K_f| \quad \text{if and only if} \quad K_f \in \mathcal{F}_y.$$

We may reproduce many of the results of section 3:

Definition 8.12. A family $\mathcal{F}$ of f-coalitions is *stable* if either it is unbalanced or $C(v_y, \mathcal{F}) \neq \emptyset$ for all $y \in D(\mathcal{F})$. ◊

Lemma 8.13. *A balanced family $\mathcal{F}$ of f-coalitions is stable if and only if*

$$z^*(v_y, \mathcal{F}) = \max\{v_y(K) \cdot y' : y' \in D(\mathcal{F})\} \le v_y(N_f) \quad \textit{for all } y \in D(\mathcal{F}). \tag{8.14}$$

The following is a variation of a part of the proof of theorem 3.5:

Lemma 8.15. *Let $\mathcal{F}$ be a balanced and stable family of f-coalitions, and let $y \in D(\mathcal{F})$. Then,*

$$v_y(N_f) = \sigma(N_f) = |N|.$$

In particular, there exists a subfamily of pairwise f-disjoint f-coalitions $\widehat{\mathcal{F}} \subset$ $\operatorname{supp}(y)$*, such that*

$$N_f = \sqcup_{F \in \widehat{F}} F.$$

Proof. We have

$$\begin{aligned}
v_y(N_f) &\geq \sum_{F\in\mathcal{F}} v_y(F) y_F && \text{by equation (8.14)} \\
&\geq \sum_{F\in\mathcal{F}} \sigma(F) y_F && \text{by definition of } v_y \\
&= \sum_{F\in\operatorname{supp}(y)} \sigma(F) y_F && \text{by definition of } \operatorname{supp}(y) \\
&= \sum_{F\in\operatorname{supp}(y)} \sum_{i\in N} a_{i,F}\, y_F && \text{by definition of } \sigma(F) \\
&= \sum_{i\in N} 1 = |N| && \text{since } y\in D(\mathcal{F}) \\
&\geq v_y(N_f) && \text{by definition of } v_y.
\end{aligned}$$

Hence, equality holds throughout. □

Theorem 8.16. *If a family $\mathcal{F}$ of f-coalitions is stable, then it is partitionable.*

Proof. If $\mathcal{F}$ is unbalanced, the result follows from the definitions, so let us assume that $\mathcal{F}$ is balanced. We have to show that if y is an extreme point of $D(\mathcal{F})$, then y is 0-1.

Given $y \in D(\mathcal{F})$, we may find a f-partition $\widehat{\mathcal{F}} \subset \mathcal{F}_y$ using lemma 8.15, and then construct the corresponding 0-1 vector $\widehat{y} \in D(\mathcal{F})$ as in equation (8.6), satisfying $\operatorname{supp}(\widehat{y}) = \widehat{\mathcal{F}}$.

Let $m \in \mathbb{Z}$, $m > 1$, be such that my is integer, and consider

$$\overline{y} = \frac{1}{m-1}\,(my - \widehat{y}).$$

Then, $\overline{y} \geq \mathbf{0}$ and $M(\mathcal{F})y = \mathbf{1}$ so that $\overline{y} \in D(\mathcal{F})$ and

$$y = \frac{1}{m}\,\widehat{y} + \frac{m-1}{m}\,\overline{y}.$$

If y is an extreme point of $D(\mathcal{F})$ we must have $y = \widehat{y}$, and hence y is 0-1. □

The equivalent to $\mathcal{W}$-functions may be defined as in 3.8:

Definition 8.17. Given a family $\mathcal{F}$ of f-coalitions, a function $v : \mathcal{F} \to \mathbb{R}$ is a *$\mathcal{W}_f$-function (for the family $\mathcal{F}$)* if

$$\sum_{F\in\widehat{\mathcal{F}}} v(F) \leq v(N_f) \quad \text{for all } f\text{-partitions } \widehat{\mathcal{F}} \subset \mathcal{F}.$$

(If there is no f-partition contained in $\mathcal{F}$, then any characteristic function is a $\mathcal{W}_f$-function.) ◊

Theorem 8.18 (cf. theorem 3.10)**.** *If the f-family $\mathcal{F}$ is partitionable, then $C(v, \mathcal{F}) \neq \emptyset$ for every $\mathcal{W}_f$-function v.*

Proof. Is similar to that of theorem 3.10. □

Theorem 8.19. *A family $\mathcal{F}$ of f-coalitions is stable if and only if it is partitionable.*

8.2 Addition of players

It is not difficult to extend some of the results in section 4 to fractional coalitions, and obtain the following, whose proofs we omit.

Lemma 8.20. *If the f-family $\mathcal{F}$ is unstable, adding a dummy player always makes the family stable and unbalanced.*

Theorem 8.21. *If $D(\mathcal{F}) \neq \emptyset$ and we add a mediator, then $\widetilde{\mathcal{F}}$ is balanced if and only if $N_f \in \mathcal{F}$.*

Let us now consider the addition of *several* players to f-coalitions in order to obtain stability when the family includes the singleton coalitions. That is, the family is of the form $\mathcal{F} = \mathcal{F}_0 \cup [N]_f$, where with $[N]_f$ we denote the family of all the f-coalitions of the form $\mathbf{e}_i$, $i = 1, \ldots, n$ (recall that these are the vectors of the canonical basis in $\mathbb{R}^n$).

Following what we did in section 5, when extending the family first we include the individual coalitions formed by the added players. Thus, if the new set of players is $\widetilde{N}$, the new family of coalitions is of the form $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_0 \cup [\widetilde{N}]_f$ and the new players are added to (some of) the coalitions in $\mathcal{F}_0$.

Unlike the case of 0-1 participation of players to coalitions, we cannot reduce the study of the polyhedron $D(\mathcal{F})$ to the study of $P(\mathcal{F}_0) = \{x \geq \mathbf{0} : M(\mathcal{F}_0)x \leq \mathbf{1}\}$, since there is not a one-to-one correspondence between the integer points in both polyhedra.

Example 8.22. Consider $N = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{F}_0 = \{F_1, F_2, F_3, F_4, F_5\}$ where

$$F_1 = (1, 0, 0, 0, 1, 1/2), \quad F_2 = (1, 1, 0, 0, 0, 1/2), \quad F_3 = (0, 1, 1, 0, 0, 1/2),$$
$$F_4 = (0, 0, 1, 1, 0, 1/2), \quad F_5 = (0, 0, 0, 1, 1, 1/2).$$

Then,

$$M(\mathcal{F}_0) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}.$$

Now, although $P(\mathcal{F}_0)$ is an integral polyhedron, $D(\mathcal{F})$ is not, since

$$(1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1/2), \ (0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1/2),$$
$$(0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 1/2), \ (0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1/2),$$
$$(0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1/2)$$

are some of its extreme points. ◊

Therefore, in order to analyze the possibility of adding players to a family of f-coalitions to obtain stability we have to study the polyhedron $D(\mathcal{F})$.

Example 8.23. Let $N = \{1, \ldots, 5\}$ and $\mathcal{K} = C_5 \cup [N]$ where

$$C_5 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\},$$

is the 5-hole.

This family is clearly unstable. Moreover, $M(C_5)$ is the clique-node matrix of the 5-hole which is not a perfect graph. From lemma 5.1 it is not possible to add a set of players and obtain a stable family preserving the set of partitions. Nevertheless, it could be reasonable to consider $P(C_5)^* = \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then, this might suggest that we add a player that participates half-time in every coalition. However, if we add this new player and its individual coalition, the resulting family of f-coalitions coincides with the family in the previous example, which was unstable. ◊

At any rate, allowing the addition of players to coalitions with fractional participation and relaxing the individual rationality of the added players, we can state a result that generalizes theorem 5.1.

Lemma 8.24. *Given an unstable family of coalitions $\mathcal{K} = \mathcal{K}_0 \cup [N]$ such that $M(\mathcal{K}_0)$ is an $n \times m$ 0-1 matrix, there is a set of players we can add with a fractional participation in the coalitions of $\mathcal{K}_0$, such that the resulting family of f-coalitions is stable.*

Proof. Since the family $\mathcal{K}$ is unstable, neither $D(\mathcal{K}) = \{(x, x') \geq \mathbf{0} : M(\mathcal{K}_0)x + Ix' = \mathbf{1}\}$ nor $P(\mathcal{K}_0)$ are integral polyhedra. Let $P(\mathcal{K}_0)^*$ be the convex hull of the integer solutions in $P(\mathcal{K}_0)$. Under the hypothesis that M is a non negative matrix, it can be proved that $P(\mathcal{K}_0)^* = \{x \geq \mathbf{0} : M(\mathcal{K}_0)x \leq \mathbf{1}, M'x \leq \mathbf{1}\}$ where $M' = (m'_{ij})_{s \times m}$ and $0 \leq m'_{ij} \leq 1$, [20].

Suppose we add the s players $i_1, \ldots, i_s$ to $\mathcal{K}_0$ in such a way player i_k participates with an amount m'_{kj} in j-th coalition, but we do not add the singleton for these new players. Let $\widetilde{\mathcal{K}}_f$ denote this new family of f-coalitions.

Then, $D(\widetilde{\mathcal{K}}_f) = \{(x, x') \geq \mathbf{0} : M(\mathcal{K}_0)x + Ix' = \mathbf{1}, M'x = \mathbf{1}\}$. Clearly, there is a one-to-one correspondence between the extreme points in $D(\widetilde{\mathcal{K}}_f)$ and the extreme points in $F = \{x \geq \mathbf{0} : M(\mathcal{K}_0)x \leq \mathbf{1}, M'x = \mathbf{1}\}$, intersection of faces of the integral polyhedron $P(\mathcal{K}_0)^*$. If it is not an empty polyhedron, and, since $M(\mathcal{K}_0)$ is 0-1 matrix, if (x, x') is an extreme point of $D(\widetilde{\mathcal{K}}_f)$, x' is also integral. Then, $D(\widetilde{\mathcal{K}}_f)$ is an integral polyhedron and the extended family of f-coalitions $\widetilde{\mathcal{K}}_f$ is stable.

If $D(\widetilde{\mathcal{K}}_f)$ is the empty set, the new family is unbalanced and stable. □

Remark 8.25. If in the previous result we arrive to an unbalanced family of coalitions, it is not convenient to add such a set of players. We could choose to add a dummy player or a mediator instead. ◊

Remark 8.26. After the addition of the players, we can arrive to unessential players. This can happen since the constraints in $P(\mathcal{K}_0)$ may not be facets of the convex hull of integral solutions $P^*(\mathcal{K}_0)$. ◊

Remark 8.27. Note that in lemma 8.24 we did not ask for individual rationality of the new players. Example 8.23 shows that this condition cannot be relaxed. ◊

Example 8.28. Let us consider again the family in example 8.23 and suppose now we don't ask for the individual rationality of the added player, then

$$M(\widetilde{F}) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that $D(\widetilde{\mathcal{F}})$ is a 0-1 polyhedron and thus $\widetilde{\mathcal{F}}$ is a stable family of f-coalitions. ♢

The result in lemma 8.24 does not hold when we consider an unstable family of f-coalitions (instead of coalitions) and add players (even without asking the individual rationality of the new players), as the following example shows.

Example 8.29. Let $\mathcal{F} = \mathcal{F}_0 \cup [N]_f$ be such that

$$M(\mathcal{F}_0) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that $\mathcal{F}$ is unstable, for instance

$$(0, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 0, 0, 0, 1/2, 0, 0, 0, 0, 0, 0, 0)$$

is in $D(\mathcal{F})$.

Following the notation in lemma 8.24, M' is the matrix having the single row $(0, 0, 0, 0, 0, 1/2, 1/2, 1/2, 1/2, 1/2)$, but after adding a new player with this participation in the set of coalitions (and without asking for her individual rationality), the extended family of f-coalitions still is unstable.

For example,

$$(0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1/2) \in D(\widetilde{\mathcal{F}}). \qquad ♢$$

8.3 Deletion of coalitions and instability indices

Let us conclude this section by making a few remarks on the deletion of f-coalitions in an unstable family in order to obtain stability, and the corresponding indices.

In the fractional case there is no correspondence between nodes in a graph and f-coalitions, but we still have that deleting f-coalitions corresponds to setting variables to zero in the polyhedron $D(\mathcal{F})$. Therefore, the following result is immediate

Lemma 8.30. *Given an unstable family of m f-coalitions $\mathcal{F}$, there is a set of k f-coalitions, $1 \leq k \leq m-1$, such that deleting them from $\mathcal{F}$ the resulting family $\widetilde{\mathcal{F}}$ is stable.*

Thus, we can extend the definition of the instability indices of a family of coalitions.

The minimum number of coalitions we have to delete in order to obtain a stable family of f-coalitions, $c(\mathcal{K}_f)$ is well defined, but, as far as we know, there is no general result for the addition of players.

Nevertheless, from lemma 8.30, we can define an instability index similar to $p(\mathcal{K})$ when $\mathcal{K} = \mathcal{K}_0 \cup [N]$ if $M(\mathcal{K}_0)$ is an $n \times m$ matrix and $G(M(\mathcal{K}_0))$ is imperfect. More precisely, we can define $p_f(\mathcal{K})$ as the minimum number of players (with fractional participation and without individual rationality) we need to add to the family to make it stable, as in lemma 8.24.

9 Appendix

In this section we present briefly some definitions and results on graph theory and polyhedral combinatorics. For further details we refer the reader to the books by Nemhauser and Wolsey [20] or Cornuéjols [9] and the references therein.

A graph G is *perfect* if and only if neither G nor its complement have an odd hole as a node induced subgraph (Strong Perfect Graph theorem [6]), where a *hole* in a graph is a chordless cycle of length at least 5.

A *clique* in a graph G is a set of nodes that are mutually adjacent in G, that is, that induces a complete subgraph of G. A clique is *maximal* if it is not contained in any other clique.

A 0-1 matrix is a *clique-node matrix* if its rows are the incidence vectors of the maximal cliques in a graph G. Conforti (see, e.g., Cornuéjols book [9]) observed that clique-node matrices can be recognized in polynomial time.

Given a 0-1 matrix M, $G(M)$ is the graph whose set of nodes is the set of columns of M, and where two nodes are adjacent if there is a row in M having a 1 in the corresponding columns.

Chvátal [8] proved that a matrix M is perfect if and only if it is the clique-node matrix of $G(M)$ and $G(M)$ is perfect.

Given a nonnegative matrix $A \in \mathbb{R}^{h \times d}$ with no zero columns, the *antiblocker* of the polyhedron $P(A) = \{x \in \mathbb{R}^d : x \geq \mathbf{0}, Ax \leq \mathbf{1}\}$, is the set of its valid inequalities, i.e.,

$$P(A)^c = \{y \in \mathbb{R}^d : y \geq \mathbf{0} \text{ and } y \cdot x \leq 1 \text{ for all } x \in P(A)\}.$$

This is a strong duality since $(P(A)^c)^c = P(A)$. If $P(B) = P(A)^c$ we say that the polyhedra $P(A)$ and $P(B)$ form an *antiblocking pair*.

If M be a 0-1 matrix without dominating rows, we may consider its *antiblocker matrix* $a(M)$, having the same number of columns as M and whose rows are the maximal 0-1 vectors y such that $y \cdot x \leq 1$ for every row x of M. It is not difficult to see that, in general, $a(a(M)) \neq M$. Moreover, equality holds if and only M is a clique-node matrix.

There is a strong relation between antiblocking pair of polyhedra, antiblocker matrices, and integer solutions. Denoting by P^* the convex hull of the integer points in the polyhedron P, the following diagram holds if M is a 0-1 matrix [10]:

$$\begin{tikzcd}
P(M) \arrow[r, phantom, "\supset"] \arrow[d, leftrightarrow] & P(a(a(M))) \arrow[r, phantom, "\supset"] \arrow[d, leftrightarrow] & P(M)^{*} \arrow[d, leftrightarrow] \\
P(M)^{c} \arrow[r, phantom, "\subset"] & P(a(M))^{*} \arrow[r, phantom, "\subset"] & P(a(M)) = (P(a(M))^{*})^{c}
\end{tikzcd} \tag{9.1}$$

where the $\updownarrow$ arrow means that the corresponding polyhedra form an antiblocking pair.

This diagram implies that if $P(a(a(M))) = P(M)^{*}$ then $P(a(M))$ is also an integral polyhedron. In this case, if $P(M)$ is not an integral polyhedron, then M is not a clique-node matrix, but $G(M)$ is perfect.

If $M(G)$ is the clique-node matrix of a graph, deleting nodes from G corresponds to setting variables to zero in the polyhedron $P(M(G))$. That is, if F is a subset of nodes of G, the integer solutions in $P(M(G \setminus F))$ are in a one to one correspondence with the integer solutions in $P(M(G))$ whose coordinates in F are zero.

References

[1] N. Aguilera, M. Escalante, G. Nasini: A Generalization of the Perfect Graph Theorem under the Disjunctive Index. *Mathematics of Operations Research* Vol 27 **3** (2002) 460–469.

[2] J.P. Aubin: Coeur et valeur des jeux flous à paiement latéraux. *C.R. Acad. Sci.* Paris 279 A (1974) 891-894.

[3] E. Boros, V. Gurvich: Perfect Graphs are kernel-solvable. *Discrete Mathematics* **159** (1996) 35–55.

[4] E. Boros, V. Gurvich, A. Vasin: Stable families of coalitions and normal hypergraphs. *Mathematical Social Sciences* **34** (1995) 22–95.

[5] R. Branzei D. Dimitrov, S. Tijs: *Models in Cooperative Game Theory Crisp, Fuzzy, and Multi-Choice Games.* Lecture Notes in Economics and Mathematical Systems, Springer (2005).

[6] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas: The Strong Perfect Graph Theorem. *Annals of Mathematics* **164** (2006) 51–229.

[7] M. Chudnovsky, G. Cornuéjols, X. Liu P. Seymour, K. Vuskovic: Recognizing Berge graphs. *Combinatorica* **25** (2006) 143–187.

[8] V. Chvátal: On certain polytopes associated with graphs. *Journal of Combinatorial Theory B* **18** (1975) 138–154.

[9] G. Cornuéjols: *Combinatorial Optimization: Packing, Covering.* CBMS SIAM (2001).

[10] D. R. Fulkerson: Antiblocking Polyhedra. *Journal of Combinatorial Theory B* **12** (1972) 50–71.

[11] S. Gerke, C. McDiarmid: Graph imperfection. *Journal of Combinatorial Theory B* **83** (2001) 58–78.

[12] V. A. Gurvich, A. A. Vasin: Reconciliable sets of coalitions. Questions in applied mathematics (Russian), Sibirsk. Ènerget. Inst., Akad. Nauk SSSR Sibirsk. Otdel., Irkutsk (1977) 20–30. English abstract: MR566169 (81b:90172).

[13] V. A. Gurvich, A. A. Vasin: Reconciliable sets of coalitions for games in normal form. Numerical Methods of Optimization (Appl. Math) (in Russian), SSSR Sibirsk. Otdel., Ènerget. Inst., Irkutsk (1978) 27–38. English abstract: MR570442 (81j:90147).

[14] C.-R. Hsiao, T. Raghavan: Shapley value for Multi-choice Cooperative Games. *Games and Economic Behavior* **5** (1993) 240–256.

[15] M. Kaneko, M. H. Wooders: Cores of partitioning games. *Mathematical Social Sciences* **3** (1982) 313–327.

[16] J. Kuipers: Combinatorial methods in cooperative game theory. Ph. D. Dissertation, Maastricht University (1994).

[17] M. Le Breton, G. Owen, S. Weber: Strongly balanced cooperative games. *International Journal of Game Theory* **21** (1992) 419–427.

[18] L. Lovász: Normal hypergraphs and the perfect graph conjecture. *Discrete Math.* **2** (1972) 253–267.

[19] A. van den Nouweland, J. Potters, S. Tijs, J. Zarzuelo: Cores and Related Solution Concepts for Multi-choice Games. *ZOR-Mathematical Methods for Operations Research* **41** (1995) 289–311.

[20] G. Nemhauser, L. Wolsey: *Integer and Combinatorial Optimization.* Wiley (1988).

[21] H. Peters, H. Zank: The Egalitarian Solution for Multichoice Games. *Annals of Operations Research* **137** (2005) 399–409.

[22] T. Quint: Necessary and sufficient conditions for balancedness in partitioning games. *Mathematical Social Sciences* **22** (1991) 87–91.

[23] S. Tijs, R. Branzei, S. Ishihara, S. Muto: On cores and stable sets for fuzzy games. *Fuzzy Sets and Systems* **146** (1991) 285–296.