THE DISJUNCTIVE PROCEDURE ON THE MATCHING PROBLEM[*]

Néstor Aguilera
CONICET and Universidad Nacional del Litoral

Graciela Nasini
Universidad Nacional de Rosario

Mariana Escalante
CONICET and Universidad Nacional de Rosario

Abstract

In this paper we study the Disjunctive Procedure on the linear relaxation of the matching polytope. We find a characterization of all the valid inequalities.We give an alternative proof of the well-known result of the integrality of the matching polytope on bipartite graphs.

1 INTRODUCTION

A Combinatorial Optimization Problem can be formulated as finding the maximum (or minimum) of a linear function on a subset K_0 of $\{0,1\}^n$. Clearly, optimizing such a function over K_0 is equivalent to do so on its convex hull, $\operatorname{conv}(K_0)$. Although in many problems K_0 can be described as

$$K_0 = \{x \in \{0,1\}^n : Ax \leq b\} \tag{1}$$

where A is an $m \times n$ real matrix and $b \in \mathbb{R}^m$, in general, it is not easy to find a description of $\operatorname{conv}(K_0)$ by linear inequalities.
A *linear relaxation* of K_0 is any polyhedron K such that $K \cap \{0,1\}^n = K_0$. If K_0 is defined as in (1) we say that K given by

$$K = \{x \in \mathbb{R}^n : 0 \leq x \leq 1 \; ; \; Ax \leq b\} = \{x \in \mathbb{R}^n : \tilde{A}x \leq \tilde{b}\}$$

is the *original linear relaxation.*
There exists a wide variety of *tightening procedures* that, beginning with a linear relaxation of K_0, find a tighter one at every step, arriving to $\operatorname{conv}(K_0)$ after a finite number of applications.

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In this paper we particularly work on the *Disjunctive Procedure*, developed by Balas, Ceria and Cornuéjols in [1] and [2].
The Disjunctive Procedure can be described as follows:
Let $K = \{x \in \mathbb{R}^n : \tilde{A}x \leq \tilde{b}\}$ be a linear relaxation of K_0. For fixed j, $1 \leq j \leq n$, every inequality in the system $\tilde{A}x \leq \tilde{b}$ is multiplied by x_j and $1 - x_j$, obtaining a system of, in general, nonlinear inequalities. Then, x_j^2 is replaced by x_j and the products $x_i x_j$, for $i \neq j$, by new variables y_i, obtaining a new linear system in the variables x and y. The polytope defined by these linear inequalities is denoted by $M_j(K)$. Finally, the polyhedron $P_j(K)$ is obtained by projecting back $M_j(K)$ onto the x-space, by eliminating the y variables.

The following result, proved in [1] and [2], gives an alternative definition for P_j:

Theorem 1.1 *For any* j, $P_j(K) = \text{conv}\{x \in K : x_j \in \{0, 1\}\}$. *In particular,* $\text{conv}(K_0) \subset P_j(K) \subset K$.

Defining, for $\{i_1, ...i_k\} \subset \{1, ..., n\}$,

$$P_{i_1 \ldots i_k}(K) = P_{i_1}(P_{i_2 \ldots i_k}(K)),$$

the following lemma characterizes the Disjunctive Procedure as a sequential tightening procedure.

Lemma 1.2 *For a polyhedron* $K \subset [0, 1]^n$ *the following assertions are true:*

1. $P_{i_1 \ldots i_k}(K) = \text{conv}\{x \in K : x_j \in \{0, 1\}$ *for any* $j \in \{i_1, ..., i_k\}\}$
2. $P_{1 \ldots n}(K) = \text{conv}(K_0)$.

Integral polyhedra in $[0, 1]^n$ can be described in terms of the Disjunctive Procedure by the following lemma

Lemma 1.3 *For a polyhedron* $K \subset [0, 1]^n$, $K = \text{conv}(K_0)$ *if and only if* $P_j(K) = K$ *for all* $j \in \{1, ..., n\}$.

Therefore, K is an integral polyhedron if and only if, for all $j \in \{1, ..., n\}$, every valid inequality of $P_j(K)$ is also valid for K_0. In general, to explicitly obtain the whole set of valid inequalities for $P_j(K)$ is an exponential task. Nevertheless, in some cases, it is possible to characterize them in terms of the particular structure of the problem. Our first goal is to obtain such a characterization by working on the matching polytope, following Ceria in his work upon the stable set polytope (see [2]).

1.1 THE MATCHING PROBLEM

Let $G = (V, E)$ be a graph with n nodes. We denote by $[i, j]$ the edge in E with extreme nodes i and j for $i, j \in V$. Given a subset U of V, $E(U)$ will denote the set of edges of G with both extreme nodes in U. If $\bar{U} \subset V$ is such that $U \cap \bar{U} = \emptyset$, then $(U : \bar{U})$ will denote the set of edges with one extreme node in U and the other one in $\bar{U}$. Finally, for each $i \in V$ we define

$$\Gamma(i) = \{j \in V : [i, j] \in E\}.$$

Definition 1.4 *Given the graph $G = (V, E)$ a matching is a subset of edges M such that in the induced subgraph $G(M) = (V, M)$ each node has at most degree* 1.

The set of all the matchings in G can be described by

$$K_0(G) = \{x \in \{0, 1\}^{|E|} : \sum_{j \in \Gamma(i)} x_{ij} \leq 1, \text{ for all } i \in V\} = \{x \in \{0, 1\}^{|E|} : Dx \leq 1\}$$

where D is the node-edge incidence matrix of G. Therefore, the original linear relaxation of $K_0(G)$ is

$$K(G) = \{x \in \mathbb{R}_+^{|E|} : Dx \leq 1\}. \tag{2}$$

In general, $K(G)$ it is not an integral polyhedron. In particular, if G contains an odd cycle $C \subset E$, the point $r \in \mathbb{R}^{|E|}$ defined as $1/2$ over C and 0 in any other case, belongs to $K(G)$. It is easy to prove that if $x \in K_0(G)$,

$$\sum_{[i,j] \in C} x_{ij} \leq \left\lfloor \frac{|C|}{2} \right\rfloor, \tag{3}$$

therefore, $r \notin \operatorname{conv}(K_0(G))$.
That means, if G is not bipartite then $K(G)$ is not an integral polyhedron. The converse is also true and it can be obtained as a corollary of the following stronger result due to Edmonds [3] which gives a description of $\operatorname{conv}(K_0(G))$.

Theorem 1.5 *The convex hull of the matchings in G is given by*

$$\begin{aligned} \sum_{j \in \Gamma(i)} x_{ij} &\leq 1 \quad \textit{for all } i \in V, \\ \sum_{[i,j] \in E(U)} x_{ij} &\leq \left\lfloor \frac{|U|}{2} \right\rfloor \quad \textit{for all } U \subset V \textit{ such that } |U| \geq 3 \textit{ odd,} \\ x &\in \mathbb{R}_+^{|E|}. \end{aligned}$$

The integrality of $K(G)$ in a bipartite graph can also be obtained as a consequence of the total unimodularity of the matrix D (see [4]). The second goal of this paper is to present an alternative proof, independent form the ones already mentioned, using the characterization of valid inequalities of $P_j\,(K(G))$.

2 CHARACTERIZATION OF VALID INEQUALITIES

Let $G = (V, E)$ and
$$K(G) = \{x \in \mathbb{R}_+^{|E|} : Dx \leq 1\}.$$
Our purpose is to find a description of the whole set of the valid inequalities for $P_{[r.s]}(K(G))$, where $P_{[r,s]}$ denotes the Disjunctive Procedure related to the variable x_{rs}, with $[r, s] \in E$.
After eliminating redundancies, $M_{[r.s]}(K(G))$ is given by the following system of linear inequalities
$$\begin{array}{rcll}
 & \sum\limits_{j\in\Gamma(i)} x_{ij} & \leq 1 & i \in V, \\
\sum\limits_{j\in\Gamma(i)} x_{ij} + x_{rs} - 1 \leq & \sum\limits_{j\in\Gamma'(i)} y_{ij} & \leq x_{rs} & i \in V' = V \setminus \{r, s\}, \\
0 \leq & y_{ij} & \leq x_{ij} & [i, j] \in E' = E \setminus \{[r, s]\},
\end{array}$$
where $\Gamma'(i) = \Gamma(i) \setminus \{r, s\}$ for every $i \in V$.
Therefore, given $x \in K(G)$, $x \in P_{[r,s]}(K(G))$ if and only if there exists $y \in \mathbb{R}^{|E'|}$ such that
$$\begin{array}{rcll}
\sum\limits_{j\in\Gamma(i)} x_{ij} + x_{rs} - 1 \leq & \sum\limits_{j\in\Gamma'(i)} y_{ij} & \leq x_{rs} & \text{for } i \in V', \\
0 \leq & y_{ij} & \leq x_{ij} & \text{for } [i, j] \in E',
\end{array}$$
or, equivalently, if and only if the system
$$\begin{array}{rcll}
 & \sum\limits_{j\in\Gamma'(i)} y_{ij} + z_i & = x_{rs} & \text{for } i \in V', \\
0 \leq & y_{ij} & \leq x_{ij} & \text{for } [i, j] \in E'' = E(V'), \\
0 \leq & z_i & \leq 1 - \sum\limits_{j\in\Gamma(i)} x_{ij} & \text{for } i \in V',
\end{array} \tag{4}$$
is feasible.
By Farkas' lemma, the system (4) has a solution if and only if the system
$$\begin{array}{rcl}
-(u_i + u_j) + v_{ij} & \geq & 0, \text{ for } [i, j] \in E'', \\
-u_i + w_i & \geq & 0, \text{ for } i \in V', \\
-\sum\limits_{i\in V'} u_i x_{rs} + \sum\limits_{[i,j]\in E'} x_{ij} v_{ij} + \sum\limits_{i\in V'} w_i (1 - \sum\limits_{j\in\Gamma(i)} x_{ij}) & < & 0, \\
v, w & \geq & 0,
\end{array} \tag{5}$$
is unfeasible.
Rewriting (5) as
$$\begin{array}{rcl}
\max(0, u_i + u_j) & \leq & v_{ij} \quad \text{for } [i, j] \in E'', \\
\max(0, u_i) & \leq & w_i \quad \text{for } i \in V', \\
-\sum\limits_{i\in V'} u_i x_{rs} + \sum\limits_{[i,j]\in E'} x_{ij} v_{ij} + \sum\limits_{i\in V'} w_i (1 - \sum\limits_{j\in\Gamma(i)} x_{ij}) & < & 0,
\end{array}$$

it is easy to prove that (5) is unfeasible if and only if there is no $u \in \mathbb{R}^{|V'|}$ such that

$$-\sum_{i\in V'} u_i x_{rs} + \sum_{[i,j]\in E''} x_{ij} \max(0, u_i + u_j) + \sum_{i\in V'} \max(0, u_i)(1 - \sum_{j\in\Gamma(i)} x_{ij}) < 0.$$

In other words, given $x \in K(G)$, $x \in P_{[r,s]}(K(G))$ if and only if, for every $u \in \mathbb{R}^{|V'|}$,

$$\sum_{i\in V'} u_i x_{rs} - \sum_{[i,j]\in E''} \max(0, u_i + u_j) x_{ij} + \sum_{i\in V'} \sum_{j\in\Gamma(i)} \max(0, u_i) x_{ij} \le \sum_{i\in V'} \max(0, u_i)$$

or, equivalently, if for every $u \in \mathbb{R}^{|V'|}$,

$$\sum_{i\in V'} u_i x_{rs} - \sum_{[i,j]\in E'',\, u_i+u_j>0} (u_i + u_j) x_{ij} + \sum_{i\in V',\, u_i>0} \sum_{j\in\Gamma(i)} u_i x_{ij} \le \sum_{i\in V',\, u_i>0} u_i. \tag{6}$$

Let us observe that each $u \in \mathbb{R}^{|V'|}$ defines a partition of V' given by

$$P = \{i \in V' : u_i > 0\} \quad \text{and} \quad \bar{P} = V' \setminus P = \{i \in V' : u_i \le 0\}.$$

Redefining u_i as $(-u_i)$ for all $i \in \bar{P}$, (6) becomes

$$\left(\sum_{i\in P} u_i - \sum_{j\in\bar{P}} u_j\right) x_{rs} - \sum_{[i,j]\in(P:\bar{P}),\, u_i-u_j>0} x_{ij}(u_i - u_j) - \sum_{[i,j]\in E(P)} x_{ij}(u_i + u_j)$$
$$+ \sum_{[i,j]\in(P:\bar{P})} u_i x_{ij} + \sum_{[i,j]\in E(P)} u_i x_{ij} + \sum_{i\in P\cap\Gamma(s)} u_i x_{is} + \sum_{i\in P\cap\Gamma(r)} u_i x_{ir} \le \sum_{i\in P} u_i$$

or equivalently

$$\left(\sum_{i\in P} u_i - \sum_{j\in\bar{P}} u_j\right) x_{rs} + \sum_{i\in P\cap\Gamma(s)} u_i x_{is} + \sum_{i\in P\cap\Gamma(r)} u_i x_{ir}$$
$$+ \sum_{[i,j]\in(P:\bar{P})} \min(u_i, u_j) x_{ij} \le \sum_{i\in P} u_i.$$

Therefore, we have proved the following theorem

Theorem 2.1 *Let D be the node-edge incidence matrix of a graph $G = (V, E)$ and $K(G) = \{x \in \mathbb{R}_+^{|E|} : Dx \le 1\}$. If $[r, s] \in E$ and $V' = V \setminus \{r, s\}$ then $x \in P_{[r,s]}(K(G))$ if and only if $x \in K(G)$ and*

$$dx_{rs} + \sum_{i\in P\cap\Gamma(s)} u_i x_{is} + \sum_{i\in P\cap\Gamma(r)} u_i x_{ir} + \sum_{[i,j]\in(P:\bar{P})} \min(u_i, u_j) x_{ij} \le \sum_{i\in P} u_i \tag{7}$$

for every $P \subset V'$ and $u \in \mathbb{R}_+^{|V'|}$, where $\bar{P} = V' \setminus P$ and $d = \sum_{i\in P} u_i - \sum_{j\in\bar{P}} u_j$.

This theorem enables us to prove that the odd cycle inequalities (3) containing the edge $[r,s]$, are valid for $P_{[r,s]}(K(G))$.

Lemma 2.2 *Given $G=(V,E)$, let $C\subset E$ be an odd cycle in G with $|C|\geq 5$ and $[r,s]\in C$. Then*

$$\sum_{[i,j]\in C} x_{ij} \leq \left\lfloor \frac{|C|}{2} \right\rfloor$$

is a valid inequality for $P_{[r,s]}(K(G))$.

Proof.
Denoting by $V(C)$ the set of nodes in C, let us color the nodes in $V'(C)=V(C)\setminus\{r,s\}$. Suppose that we begin by coloring the node next to r in C (different form s) in red, and we color alternatively, in blue and red until the one next to s (different form r).
Let P denote the set of the red nodes and $u\in\mathbb{R}_+^{|V'|}$ given by

$$u_i = \begin{cases} 1 & \text{if } i\in V'(C), \\ 0 & \text{in any other case.} \end{cases} \tag{8}$$

The inequality (7) related to P and u becomes

$$x_{rs} + \sum_{i\in P\cap\Gamma(s)} x_{is} + \sum_{i\in P\cap\Gamma(r)} x_{ir} + \sum_{[i,j]\in(P:\bar{P})} x_{ij} \leq |P| = \left\lfloor \frac{|C|}{2} \right\rfloor$$

which is a valid inequality for $P_{[r,s]}(K(G))$ and it is clearly stronger than

$$\sum_{[i,j]\in E(C)} x_{ij} \leq \left\lfloor \frac{|C|}{2} \right\rfloor .$$

■
We close this section by eliminating redundancies in the description of valid inequalities given by (7).

Remark 2.3 *Given $P\subset V'$ and $u\in\mathbb{R}_+^{|V'|}$ such that $d\leq 0$, the inequality (7) associated to P and u is dominated by*

$$\sum_{i\in P\cap\Gamma(s)} u_i x_{is} + \sum_{i\in P\cap\Gamma(r)} u_i x_{ir} + \sum_{[i,j]\in(P:\bar{P})} u_i x_{ij} \leq \sum_{i\in P} u_i$$

which is a non-negative linear combination of the inequalities defining $K(G)$.

Remark 2.4 *Let $i\in V'$, $P=\{i\}$ and u given by*

$$u_k = \begin{cases} 1 & \text{if } \; k=i, \\ 0 & \text{if } \; k\neq i. \end{cases}$$

If $U_i=\{i,r,s\}$ then the inequality (7) associated to P and u is

$$\sum_{[k,l]\in E(U_i)} x_{kl} \leq 1. \tag{9}$$

Remark 2.5 *Let $P \subset V'$ and $u \in \mathbb{R}_+^{|V'|}$ such that $u_j = 0$ for every $j \in \bar{P}$, then the corresponding inequality (7) becomes*

$$\sum_{i\in P} u_i x_{rs} + \sum_{i\in P\cap\Gamma(s)} u_i x_{is} + \sum_{i\in P\cap\Gamma(r)} u_i x_{ir} \leq \sum_{i\in P} u_i$$

or

$$\sum_{i\in P} u_i \sum_{[k,l]\in E(U_i)} x_{kl} \leq \sum_{i\in P} u_i$$

which is a non-negative linear combination of the inequalities (9).

Taking into account Remarks 2.3, 2.4 and 2.5, in the description of $P_{[r,s]}(K(G))$ we need to consider the inequalities defining $K(G)$, the ones of the form (9), and the inequalities (7) such that $d > 0$ and $\sum\limits_{i\in\bar{P}} u_j > 0$.

3 MATCHING POLYTOPE ON BIPARTITE GRAPHS

In Section 1 we have mentioned some well-known proofs of the integrality of $K(G)$ when G is a bipartite graph. In this section we give an alternative proof of this result using Theorem 2.1.

Theorem 3.1 *If G is a bipartite graph then $K(G) = \mathrm{conv}\,(K_0(G))$.*

Proof.
Let $G = (V, E)$ be a bipartite graph. If $|V| \leq 3$ then the proof is obvious. Therefore we prove the general result by induction on $|V|$. Let us suppose that, for every bipartite graph G with $|V| \leq k$, $K(G) = \mathrm{conv}\,(K_0(G))$.
Let G be a bipartite graph such that $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$, $E \subset V_1 \times V_2$ and $|V| = k + 1 \geq 4$. We will prove that for every $[r, s] \in E$, any valid inequality for $P_{[r,s]}(K(G))$ is also valid for $K(G)$.
From now on we make use of the notation introduced in the previous section. Let $[r, s] \in E$ with $r \in V_1$ and $s \in V_2$.
It is clear that if G is bipartite, the inequalities (9) can be obtained as a linear non-negative combination of the inequalities defining $K(G)$. Therefore we only need to prove that for every $P \subset V'$ and $u \in \mathbb{R}_+^{|V'|}$ such that $d > 0$ and $\sum\limits_{j\in\bar{P}} u_j > 0$,

$$dx_{rs} + \sum_{i\in P\cap\Gamma(s)} u_i x_{is} + \sum_{i\in P\cap\Gamma(r)} u_i x_{ir} + \sum_{[i,j]\in(P:\bar{P})} \min(u_i, u_j) x_{ij} \leq \sum_{i\in P} u_i,$$

is a valid inequality for $K(G)$.
Let us first consider the case $P = V_1 \setminus \{r\}$ and $\bar{P} = V_2 \setminus \{s\}$, where (7) becomes

$$dx_{rs} + \sum_{i\in P\cap\Gamma(s)} u_i x_{is} + \sum_{[i,j]\in(P:\bar{P})} \min(u_i, u_j) x_{ij} \leq \sum_{i\in P} u_i. \tag{10}$$

Let us observe that if $x \in K(G)$ and

$$\sum_{i \in P \cap \Gamma(s)} (u_i - d)x_{is} + \sum_{[i,j] \in (P:\bar{P})} \min(u_i, u_j)x_{ij} \leq \sum_{j \in \bar{P}} u_j$$

then x satisfies (10) because

$$\begin{aligned}
& dx_{rs} + \sum_{i \in P \cap \Gamma(s)} u_i x_{is} + \sum_{[i,j] \in (P:\bar{P})} \min(u_i, u_j)x_{ij} \\
& \leq d(1 - \sum_{i \in P \cap \Gamma(s)} x_{is}) + \sum_{i \in P} u_i x_{is} + \sum_{[i,j] \in (P:\bar{P})} \min(u_i, u_j)x_{ij} \\
& = d + \sum_{i \in P \cap \Gamma(s)} (u_i - d)x_{is} + \sum_{[i,j] \in (P:\bar{P})} \min(u_i, u_j)x_{ij} \\
& \leq d + \sum_{j \in \bar{P}} u_j = \sum_{i \in P} u_i.
\end{aligned}$$

Therefore, it is enough to prove that for every $x \in K(G)$,

$$\sum_{i \in P} (u_i - d)x_{is} + \sum_{[i,j] \in (P:\bar{P})} \min(u_i, u_j)x_{ij} \leq \sum_{j \in \bar{P}} u_j. \tag{11}$$

If $\tilde{G} = (\tilde{V}, \tilde{E})$ is the subgraph of G induced by $\tilde{V} = V \setminus \{r\}$, let us observe that the variables with positive coefficients in (11) are variables associated to arcs in $\tilde{E}$. Let $x \in K(G)$ and we define $\tilde{x} \in \mathbb{R}^{|\tilde{E}|}$ such that $\tilde{x}_{ij} = x_{ij}$ for every $[i,j] \in \tilde{E}$. Clearly we have that $\tilde{x} \in K(\tilde{G})$. Since $\tilde{G}$ is a bipartite graph with $\left|\tilde{V}\right| = k$, by inductive hypothesis, $\tilde{x} \in \mathrm{conv}(K_0(G))$. Therefore, to prove that $x \in K(G)$ satisfies (11) it is enough to do it for every $x \in K(\tilde{G})$ such that $x \in \{0,1\}^{|\tilde{E}|}$.
Let $x \in K(\tilde{G}) \cap \{0,1\}^{|\tilde{E}|}$.

- If $x_{is} = 0$ for all $i \in P$ then

$$\begin{aligned}
& \sum_{i \in P} (u_i - d)x_{is} + \sum_{[i,j] \in (P:\bar{P})} \min(u_i, u_j)x_{ij} \\
= & \sum_{[i,j] \in (P:\bar{P})} \min(u_i, u_j)x_{ij} \\
\leq & \sum_{j \in \bar{P}} u_j \sum_{i \in P} x_{ij} \leq \sum_{j \in \bar{P}} u_j.
\end{aligned}$$

- If $x_{ks} = 1$ for some $k \in P$, then $x_{is} = 0$ for every $i \in P \setminus \{k\}$ and $x_{kj} = 0$ for every $j \in \bar{P}$. Therefore

$$\begin{aligned}
&\sum_{i \in P}(u_i - d)x_{is} + \sum_{[i,j]\in(P:\bar{P})} \min(u_i, u_j)x_{ij} \\
&= u_k - d + \sum_{i \in P\setminus\{k\}} \sum_{j \in \bar{P}} \min(u_i, u_j)x_{ij} \\
&\leq u_k - d + \sum_{i \in P\setminus\{k\}} u_i \sum_{j \in \bar{P}} x_{ij} \\
&\leq \sum_{i \in P} u_i - d = \sum_{j \in \bar{P}} u_j.
\end{aligned}$$

In this way we have proved that for $P = V_1 \setminus \{r\}$, (7) is a valid inequality for $K(G)$, for all $u \in \mathbb{R}_+^{|V'|}$.
Now, suppose that P is any subset of V', $\bar{P} = V' \setminus P$ and $u \in \mathbb{R}_+^{|V'|}$.
Let us define

$$\begin{aligned}
P_1 = P \cap V_1, &\quad P_2 = P \cap V_2 \\
\bar{P}_1 = \bar{P} \cap V_1, &\quad \bar{P}_2 = \bar{P} \cap V_2.
\end{aligned}$$

Moreover, let us consider u' and u'' given by

$$u'_i = \begin{cases} u_i & \text{if } i \in P_1 \cup \bar{P}_2, \\ \\ 0 & \text{in any other case,} \end{cases} \qquad u''_i = \begin{cases} u_i & \text{if } i \in \bar{P}_1 \cup P_2, \\ \\ 0 & \text{in any other case.} \end{cases} .$$

From the previous case, if $P' = V_1 \setminus \{r\}$ the inequalities (7) associated to P', u' and u'', are valid for $K(G)$. Therefore, for every $x \in K(G)$ we have that

$$\left(\sum_{i \in P_1} u_i - \sum_{j \in \bar{P}_2} u_j\right) x_{rs} + \sum_{i \in P_1 \cap \Gamma(s)} u_i x_{is} + \sum_{[i,j]\in(P_1:\bar{P}_2)} \min(u_i, u_j)x_{ij} \leq \sum_{i \in P_1} u_i \tag{12}$$

and

$$\left(\sum_{i \in P_2} u_i - \sum_{j \in \bar{P}_1} u_j\right) x_{rs} + \sum_{i \in P_2 \cap \Gamma(r)} u_i x_{ir} + \sum_{[i,j]\in(P_2:\bar{P}_1)} \min(u_i, u_j)x_{ij} \leq \sum_{i \in P_2} u_i. \tag{13}$$

Clearly, the sum of (12) and (13) gives

$$d x_{rs} + \sum_{i \in P_1 \cap \Gamma(s)} u_i x_{is} + \sum_{i \in P_2 \cap \Gamma(r)} u_i x_{ir} + \sum_{[i,j]\in(P:\bar{P})} \min(u_i, u_j)x_{ij} \leq \sum_{i \in P} u_i,$$

a valid inequality for $K(G)$. ■

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