Notes on "Ideal 0, 1 Matrices" by Cornuéjols and Novick

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Abstract

In 1994, Cornuéjols and Novick published a classification of ideal and minimally non ideal circulant clutters. One of their main results for doing so relates contractions of these clutters, simple directed cycles in an appropriate graph, and algebraic conditions. The purpose of this paper is twofold: to correct a small inaccuracy of the necessity of the algebraic conditions in the original proof, and to show that these algebraic conditions are actually sufficient, by giving a constructive proof of the existence of cycles.

1 Introduction

Cornuéjols and Novick [1] described many ideal and minimally non clutters, studying in particular the circulant clutters $\mathscr{C}_n^k$ which are ideal or minimally non ideal (we refer the reader to the next subsection for basic notations and definitions).

One of the main tools in their classification is the following lemma:

1.1 Lemma (lemma 4.5 in [1])**.** *Suppose $2 \leq k \leq n-2$. If a subset N of $V(\mathscr{C}_n^k)$ induces a simple directed cycle, D, in $G(\mathscr{C}_n^k)$, then there exists n_1, n_2, $n_3 \in \mathbb{Z}_+$, $n_1 \geq 1$, such that*

(i) $nn_1 = kn_2 + (k+1)\, n_3$,

(ii) $\gcd(n_1, n_2, n_3) = 1$,

(iii) If $k - n_1 \leq 0$, then $E(\mathscr{C}_n^k/N) = \emptyset$ or $\{\emptyset\}$. If $k - n_1 \geq 1$, then $\mathscr{C}_n^k/N$ is of the form $\mathscr{C}_{n-n_2-n_3}^{k-n_1}$.

There seems to be an inaccuracy in the proof given in [1] to show that (ii) holds: it is stated there that $n_2 + n_3 \equiv -1 \pmod{n_1}$ and this need not be true in general, as shown by the following example.

1.2 Example. Consider

$$n = 11, \quad k = 8, \quad n_1 = 7, \quad n_2 = 4, \quad n_3 = 5,$$

and the cycle $(0, 8, 5, 3, 1, 9, 6, 4, 2, 0)$, with increments $(8, 8, 9, 9, 8, 8, 9, 9, 9)$. $\Diamond$

In section 2 we show first that if D is a simple directed cycle in $G(\mathscr{C}_n^k)$, then some algebraic conditions must be satisfied, and give afterwards a proof of the property $\gcd(n_1, n_2, n_3) = 1$.

In the final section we show that the given algebraic conditions are actually sufficient for the existence of such cycles. We point out that this construction is not needed for Cornuéjols and Novicks's results on ideal or minimally non ideal circulant clutters.

1.1 Basic notations and definitions

We will follow mostly the notations and definitions used by Cornuéjols and Novick, except for a few instances which will be indicated here. For further notations and definitions, we refer the reader to the original article [1].

$\mathbb{Z}_n$ denotes the set of equivalence classes of the integers modulo n, which in this paper will always be represented by $\{0, \ldots, n-1\}$. For fixed $k, n \in \mathbb{N}$, $1 \leq k \leq n-1$, we set $C_i = \{i, i+1, \ldots, i+(k-1)\}$ (sums taken modulo n) and define the *circulant clutter* $\mathscr{C}_n^k$ by setting $V(\mathscr{C}_n^k) = \mathbb{Z}_n$ and $E(\mathscr{C}_n^k) = \{C_i : i \in \mathbb{Z}_n\}$. $G(\mathscr{C}_n^k)$ denotes the directed graph having vertex set $V(\mathscr{C}_n^k) = \mathbb{Z}_n$, and (i, i') is an arc of $G(\mathscr{C}_n^k)$ if and only if either $i' = i + k \pmod{n}$ or $i' = i + (k+1) \pmod{n}$. A simple directed cycle in $G(\mathscr{C}_n^k)$ is a sequence of vertices $(v_0, v_1, \ldots, v_m = v_0)$, with $v_0, v_1, \ldots, v_{m-1}$ distinct, such that (v_i, v_{i+1}) is an arc of $G(\mathscr{C}_n^k)$; and we allow $m = 1$, so that a loop will be considered to be a simple directed cycle.

2 Properties of cycles in $G(\mathscr{C}_n^k)$

Suppose D is a simple directed cycle in $G(\mathscr{C}_n^k)$, and let $N = V(D)$, n_2 be the number of arcs of length k in D, and n_3 be the number of arcs or length $k+1$. Since D is a simple directed cycle, $n_2 k + n_3 (k+1)$ is a multiple of n, and therefore there exists a unique n_1 such that

$$n_1 n = n_2 k + n_3 (k+1). \tag{2.1}$$

For fixed n_1 the general solution for the unknowns n_2 and n_3 of this diophantine equation is given by

$$n_2 = -n_1 n + z (k+1) \quad \text{and} \quad n_3 = n_1 n - zk \qquad \text{for any } z \in \mathbb{Z},$$

Adding these equations for n_2 and n_3 we obtain $n_2 + n_3 = z$. On the other hand, if $m = |N|$, since D is simple, we have

$$m = n_2 + n_3, \tag{2.2}$$

and therefore,

$$n_2 = -n_1 n + m\,(k+1), \quad n_3 = n_1 n - mk. \tag{2.3}$$

Thus, given n_2 and n_3 we may obtain m and n_1 by means of the equations (2.2) and (2.1), and, conversely, given m and n_1 we may obtain n_2 and n_3 by means of the equations in (2.3).

It is rather easy to show now:

2.4 Lemma. *If m, n_1, n_2 and n_3 are nonnegative integers satisfying the equations* (2.1) *and* (2.2) *(hence also* (2.3)*), then* $\gcd(n_1, n_2, n_3) = \gcd(n_1, m)$.

Suppose that m, n_1, n_2 and n_3 are non negative integers so that the equations (2.1) and (2.2) hold. Then, $mk \le n_1 n \le m\,(k+1)$, or

$$\frac{k}{n} \le \frac{n_1}{m} \le \frac{k+1}{n}. \tag{2.5}$$

Since we always have $\lceil km/n \rceil \ge \lfloor (k+1)\,m/n \rfloor$, we may state:

2.6 Lemma. *Let n, k, m be given, with $1 \le k \le n-1$ and $0 \le m \le n-1$. Then there exist $n_1, n_2, n_3 \ge 0$ satisfying the equations* (2.1) *and* (2.2) *if and only if*

$$\left\lceil \frac{km}{n} \right\rceil = \left\lfloor \frac{(k+1)\,m}{n} \right\rfloor. \tag{2.7}$$

Moreover, if the equality (2.7) *holds, n_1 is determined by*

$$n_1 = \left\lceil \frac{km}{n} \right\rceil, \tag{2.8}$$

and

(i) $m = 0$ if and only if $n_1 = 0$, and if $m > 0$ then $0 < n_1 \le \min\{m, k\}$.

(ii) n_2 and n_3 are uniquely determined by the equations in (2.3).

2.9 Lemma. *If the assumptions of lemma 1.1 hold, then* $\gcd(m, n_1) = 1$.

Proof. Let $d = \gcd(m, n_1)$, and suppose $D = (v_0, v_1, \ldots, v_m = v_0)$. Let $\delta_i \in \{k, k+1\}$ be defined by $v_{i+1} = v_i + \delta_i \pmod{n}$ for $i = 0, \ldots, m-1$, and therefore $|\{i : \delta_i = k\}| = n_2$ and $|\{i : \delta_i = k+1\}| = n_3$.

Suppose that $d > 1$, and consider

$$n_1' = n_1/d, \quad n_2' = n_2/d, \quad n_3' = n_3/d, \quad m' = m/d,$$

so that $n_1' n = n_2' k + n_3'\,(k+1)$, and $m' = n_2' + n_3'$. For $j = 0, \ldots, m - m'$, let us define $s_j = |\{i : j \le i < j + m',\ \delta_i = k\}|$.

If there exists j, $0 \le j \le m - m'$, such that $s_j = n_2'$, then

$$v_{j+m'} \equiv v_j + \delta_j + \cdots + \delta_{j+m'-1} \equiv v_j + n_2' k + n_3'\,(k+1) \equiv v_j \pmod{n},$$

and the cycle is not simple. Suppose now that $s_j \neq n_2'$ for all j, and consider the sums $s_0, s_{m'}, s_{2m'}, \ldots, s_{m-m'} = s_{(d-1)\,m'}$. We cannot have $s_{im'} > n_2'$ for all $i = 0, \ldots, d-1$, since this would imply

$$n_2 = |\{i : \delta_i = k\}| = \sum_{j=0}^{d-1} s_{jm'} > dn_2' = n_2.$$

Similarly, we cannot have $s_{im'} < n_2'$ for all $i = 0, \ldots, d-1$. That is, there exists i such that one of $\{s_{im'}, s_{(i+1)m'}\}$ is greater than n_2' and the other is smaller. Since $s_{j+1} - s_j \in \{0, 1, -1\}$, and the values $s_{im'}, s_{im'+1}, \ldots, s_{(i+1)\,m'}$ go from something smaller than n_2' to something greater, or vice versa, we must have

$$s_j = n_2' \quad \text{for some } j \in \{im' + 1, \ldots, (i+1)\,m' - 1\},$$

which is a contradiction. □

3 Existence of cycles in $G(\mathscr{C}_n^k)$

We will show now that the algebraic conditions of the previous section are also sufficient for the existence of a simple directed cycle:

3.1 Theorem. *Let n, k and m be given, $1 \leq k \leq n-1$, $1 \leq m \leq n-1$. Then there exists a simple directed cycle D in $G(\mathscr{C}_n^k)$ with $|V(D)| = m$ if and only if*

1. *equation (2.7) holds, and*
2. *if n_1 is defined as in equation (2.8), then $\gcd(m, n_1) = 1$.*

We will split the proof of this theorem into several steps, noticing that the "only if" part is covered by lemmas 2.6 and 2.9.

Let us suppose now that the inequalities in (2.5) hold. Defining the values n_2 and n_3 as in the equations (2.3), we seek a simple directed cycle with n_2 arcs of length k, and n_3 arcs of length $k+1$ in $G(\mathscr{C}_n^k)$. Since we are assuming $V(G(\mathscr{C}_n^k)) = V(\mathscr{C}_n^k) = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, it will be enough to construct a cycle through $0 \in \mathbb{Z}_n$.

If $k/n = n_1/m$, then $n_3 = 0$ and $n_2 = m$. Since $\gcd(m, n_1) = 1$, we must have $n = dm$ and $k = dn_1$, where $d = \gcd(n, k)$. Thus, $(0, k, 2k, \ldots, (mk) = 0)$ (products taken modulo n) is a simple directed cycle. Similarly, if $(k+1)/n = n_1/m$, then $n_2 = 0$, $n_3 = m$, and we may construct the simple directed cycle $(0, k+1, 2\,(k+1), \ldots, (m\,(k+1)) = 0)$ (sums and products modulo n).

Let us assume now that the inequalities in (2.5) are strict, and therefore

$$1 \leq n_1 < m, \quad n_2, n_3 > 0.$$

Although we know what number of arcs of length k or $k+1$ to include, not any order will make a simple directed cycle:

3.2 Example. If in the example (1.2) we take the increments in the order $(8, 8, 8, 9, 8, 9, 9, 9, 9)$, we obtain the cycle $(0, 8, 5, 2, 0, 8, 6, 4, 2, 0)$, which is not simple. ♢

To construct a simple directed cycle, we will construct a (simple directed) path $(P_0, P_1, \ldots, P_m)$, with end points $P_0 = (0, 0)$ and $P_m = (n_2 k, n_3 (k + 1))$, in the lattice $\{rk, s (k + 1) : r, s \in \mathbb{Z}\}$, and moving only rightwards or upwards. Thus, the path will remain inside the rectangle

$$R_0 = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le n_2 k,\ 0 \le y \le n_3 (k + 1)\}.$$

Once an appropriate path has been constructed in the restricted lattice

$$\mathcal{R} = \{(rk, s (k + 1)) : r, s \in \mathbb{Z},\ 0 \le r \le n_2,\ 0 \le s \le n_3\},$$

the simple directed path in $G(\mathscr{C}_n^k)$ will be obtained by taking

$$v_j = a_j + b_j \pmod n \quad \text{if } P_j = (a_j, b_j), \qquad \text{for } j = 0, \ldots, m.$$

Except at its end points, our path will have to avoid points in the restricted lattice $\mathcal{R}$ of the form (a, b) with $a + b \equiv 0 \pmod n$. That is, there is a second restricted lattice involved:

$$\mathcal{M} = \{P = (a, b) \in \mathcal{R} : a + b \equiv 0 \pmod n\}.$$

If $\gcd(m, n_1) = 1$, then—as we will show momentarily—there are no points of $\mathcal{M}$ in the interior of the diagonal ℓ of R_0 joining the vertices $(0, 0)$ and $(n_2 k, n_3 (k + 1))$, and our aim is to construct a path staying as close as possible to ℓ.

3.3 Example. To fix ideas, we have taken the values $n = 70$, $k = 10$, $m = 47$, $n_1 = 7$, $n_2 = 27$, $n_3 = 20$, and sketched in figure 3.1 the points of $\mathcal{M}$ as black dots, and the diagonal ℓ of the rectangle R_0 with a thick trace. The underlying grid is the restricted lattice $\mathcal{R}$.

Let us denote by λ the slope of ℓ,

$$\lambda = \frac{n_3 (k + 1)}{n_2 k},$$

and for $P = (rk, s (k + 1)) \in \mathcal{M}$ let us define h and t by

$$h = r + s, \qquad tn = rk + s (k + 1). \tag{3.4a}$$

From the equations (2.3) applied to h and t, we know that

$$r = -tn + h (k + 1), \quad s = tn - hk. \tag{3.4b}$$

If $P \in \mathcal{M} \cap \ell$, and $P \ne (0, 0)$, we have

$$\lambda = \frac{n_3 (k + 1)}{n_2 k} = \frac{s (k + 1)}{rk},$$

which written in terms of m, n_1, h and t, yields $tm = n_1 h$. Since $\gcd(m, n_1) = 1$ and we are assuming $P \in R_0$, this implies $t = n_1$, $h = m$. Thus:

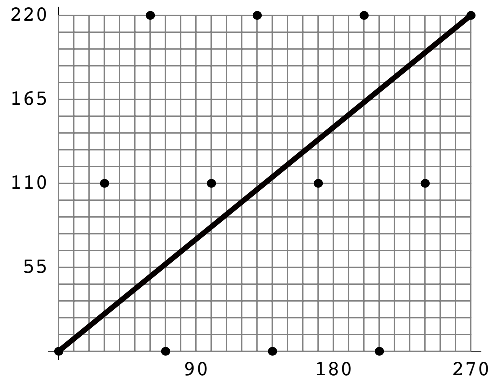

Figure 3.1: The lattice $\mathcal{M}$ and the diagonal ℓ.

3.5 Lemma. *If* $\gcd(m, n_1) = 1$*, then the only points in* $\mathcal{M} \cap \ell$ *are the end points of* ℓ*.*

Using the same type of calculations, we may estimate the distance from a point $P \in \mathcal{M}$ to ℓ. To be more precise, for $P = (x, y) \in R_0$, let $\delta_x(P)$ and $\delta_y(P)$ denote the horizontal and vertical distances from P to ℓ, i.e.,

$$\delta_x(P) = |x - y/\lambda\ |, \quad \delta_y(P) = |y - \lambda x|.$$

Then, if $P \in \mathcal{M}$ is above ℓ, with the notations in (3.4), we have

$$\delta_y(P) = s\,(k+1) - \lambda r k = \frac{n}{n_2}\,(tm - n_1 h)\,(k+1).$$

Since $tm - n_1 h$ is a positive integer, we must have $tm - n_1 h \geq 1$, so that

$$\delta_y(P) \geq \frac{n}{n_2}\,(k+1).$$

Given that $\delta_x(P) = \delta_y(P)/\lambda$, we also have

$$\delta_x(P) = \frac{n_2\,k}{n_3\,(k+1)}\,\delta_y(P) \geq \frac{n}{n_3}\,k.$$

Using similar arguments when $P \in \mathcal{M}$ is below ℓ (or just using symmetry about the midpoint of ℓ), we get:

3.6 Lemma. *If* $P = (a, b) = (rk, s\,(k+1)) \in \mathcal{M} - \ell$*, then*

$$\delta_x(P) \geq \frac{n}{n_3}\,k \quad \textit{and} \quad \delta_y(P) \geq \frac{n}{n_2}\,(k+1).$$

We define the path $(P_0, \ldots, P_m)$ in $\mathcal{R}$, with $P_j = (a_j, b_j)$ recursively by

$$P_0 = (0, 0),$$

and for $j = 1, \ldots, m$,

$$P_j = (a_j, b_j) = \begin{cases} P_{j-1} + (k, 0) & \text{if } b_{j-1} \geq \lambda a_{j-1}, \\ P_{j-1} + (0, k+1) & \text{in other case.} \end{cases}$$

In words: if P_j is on or above ℓ we move one step (of length k) to the right, or else we move up one step (of length $k+1$). It should be clear that in the path $(P_0, \ldots, P_m)$ there is one direction (up or to the right, depending on whether $n_3 \leq n_2$ or not), in which we never make two consecutive steps. More formally:

3.7 Lemma. *Let $(P_0, \ldots, P_m)$ be the path defined previously. Then, for $j = 0, \ldots, m$, we have:*

(i) $\delta_y(P_i) \leq k+1$ *if* $n_3 \leq n_2$,

(ii) $\delta_x(P_i) \leq k$ *if* $n_2 \leq n_3$.

Moreover,

(iii) if $n_3 \leq n_2$ *and* P_j *is below* ℓ, *then* $\delta_y(P_j) \leq n_3\,(k+1)/n_2$,

(iv) if $n_2 \leq n_3$ *and* P_j *is above* ℓ, *then* $\delta_x(P_j) \leq n_2\,k/n_3$.

We will show now that we cannot have two distinct points, $P_j = (a_j, b_j)$ and $P_{j'} = (a_{j'}, b_{j'})$, in the path with $a_j + b_j \equiv a_{j'} + b_{j'} \pmod{n}$, unless they are the end points of ℓ. Suppose, by contradiction, that this is true for some $j < j'$, so that $P_{j'} \in P_j + \mathcal{M} = \{P_j + P : P \in \mathcal{M}\}$.

Let us assume $n_3 \leq n_2$ (the case $n_3 > n_2$ is similar), and consider the point Q on $P_j + \ell$ having the same horizontal coordinate as $P_{j'}$ (Q is not necessarily inside the rectangle R_0). By lemma 3.6, the vertical distance $\delta_y(P_{j'}, Q)$ between $P_{j'} \in P_j + \mathcal{M}$ and $Q \in P_j + \ell$ satisfies

$$\delta_y(P_{j'}, Q) \geq \frac{n}{n_2}\,(k+1),$$

unless $P_j = (0,0)$ and $P_{j'} = (n_2 k, n_3\,(k+1))$.

Noticing that $\delta_y(P_j) = \delta_y(Q)$, since the segment $P_j Q$ is parallel to ℓ, we see that if $P_{j'}$ and Q are on the same side of ℓ, by lemma 3.7 we have

$$\frac{n}{n_2}\,(k+1) \leq \delta_y(P_{j'}, Q) \leq \delta_y P_{j'} \leq k+1,$$

which is impossible since $n_2 < n$. On the other hand, if $P_{j'}$ and Q are on different sides of ℓ, applying the second part of lemma 3.7 to either P_j or $P_{j'}$, we obtain

$$\frac{n}{n_2}\,(k+1) \leq \delta_y(P_{j'}, Q) \leq \delta_y(P_{j'}) + \delta_y(P_j) \leq \frac{n_3}{n_2}\,(k+1) + k + 1 = \frac{m}{n_2}\,(k+1),$$

again a contradiction since $m < n$. Thus, theorem 3.1 is proved.

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References

[1] G. Cornuéjols, B. Novick, Ideal 0, 1 Matrices, J. Combin. Theory Ser. B 60 (1994) 145–157.