Characterizations of Postman Sets

Néstor E. Aguilera Valeria A. Leoni

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Abstract

Using results by McKee and Woodall on binary matroids, we prove that the set of postman sets has odd cardinality, generalizing a result by Toida on the cardinality of cycles in Eulerian graphs. We study the relationship between T-joins and blocks of the underlying graph, obtaining a decomposition of postman sets in terms of blocks. We conclude by giving several characterizations of T-joins which are postman sets and commenting on practical issues.

Keywords: T-joins, postman sets, cardinality, graph block, decomposition

1 Basic Notations and Definitions

We will consider undirected graphs $G = (V, E)$. The set of odd degree vertices of G will be denoted by $O(G)$, or simply by O when it is clear what the underlying graph is. Most other notations and conventions for graphs are similar to those in West [5]. In particular, paths and cycles have no repeated vertices, and loops are cycles.

Given a subset T of vertices with $|T|$ even, a set of edges $J \subset E$ is a *T-join* if $O(G_J) = T$ where $G_J = (V, J)$. We will be interested in the family $\mathcal{T}$ of minimal T-joins: an inclusion-wise minimal T-join is just a T-join such that G_J is acyclic. Of course, $\mathcal{T}$ is a *clutter*, i.e. a family of subsets of some finite base set—here E—none of which is included in another.

When $T = \emptyset$, the empty set is the unique minimal $\emptyset$-join, and it is convenient to work instead with the clutter of cycles (regarded as edge-sets) $\mathcal{C}$, so that every non-empty $\emptyset$-join may be written as a union of disjoint cycles. When $T = O(G)$, the minimal T-joins are called *postman sets*, and we will indicate the corresponding clutter by $\mathcal{P}$.

We observe that although there are always postman sets, perhaps only the empty set (i.e. $\mathcal{P} = \{\emptyset\}$), we may have $\mathcal{T} = \emptyset$ if some connected component of G contains an odd number of vertices of T. Similarly, $\mathcal{C}$ could be empty.

2 Introduction

In 1973, S. Toida [4] proved that in an Eulerian graph there is an odd number of cycles passing through any given edge. This can be shown by deleting the edge, say with endpoints u and v, from the graph and showing that there is an odd number of (simple) u, v-paths in the resulting graph G'. In this case $O(G') = \{u, v\}$, and the u, v-paths in G' are precisely the postman sets in G'.

T. McKee [2] showed in 1984 that Toida's result actually characterizes Eulerian graphs: every edge is in an odd number of cycles if and only if $O(G) = \emptyset$. It is worth mentioning that in 1990, D. Woodall [6] gave an alternative proof of McKee's converse, and both McKee and Woodall obtained it as a consequence of more general results in the frame of binary matroids, which we reproduce here as Theorem 3.3.

We use McKee's and Woodall's results directly to show a characterization of the family of postman sets through a condition involving all minimal T-joins and cycles, the precise statement being given in Corollary 3.4. As a consequence of this characterization, in Corollary 3.5 we generalize Toida's result to postman sets in any graph, obtaining that $\mathcal{P}$ has odd cardinality.

Although to prove this extension we rely on the McKee's and Woodall's results, it could also be proved inside graph theory (without explicit mention of binary matroids), for example by induction on the number of edges (see also the Remark after Theorem 5.1).

Certainly, for general T-joins it is not true that $|\mathcal{T}|$ is odd. For instance, in the kite of Figure 1, there are four T-joins when $T = \{2, 4\}$.

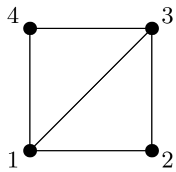
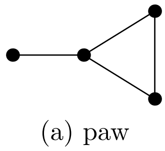


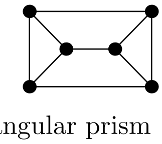
Figure 1: kite

In view of McKee's result, it is natural to wonder whether $|\mathcal{T}|$ odd implies $T = O$. However, this is not true. For example, in the kite of Figure 1 with $T = \{1, 2\}$ we have $|\mathcal{T}| = 3$, but $T \not\subset O$ and $O \not\subset T$.

A simple way of looking at McKee's converse of Toida's result is to consider the symmetric difference of all cycles. Similarly, E will be itself a T-join (see Lemma 3.1 below) and therefore $T = O$ if every edge is in an odd number of minimal T-joins. However, even for postman sets not always do we have the latter property. For example, in the paw of Figure 2(a), the edges in the cycle do not belong to any postman set. And in the triangular prism (the cartesian product $K_2 \,\square\, C_3$) shown in Figure 2(b), the edges in the triangular bases belong to an even number of postman sets, or equivalently, those edges which are not in any of the triangular bases form a postman set and intersect an even number of cycles.



(a) paw



(b) triangular prism ($K_2 \square C_3$)

Figure 2: paw and triangular prism

As the example of the paw suggests, the blocks of the graph play an important role in the structure of T-joins and postman sets, and we study this interplay in Section 4. Lemma 4.2, on the intersection of clutters, is the key for studying this relationship, and allows us to show in Theorem 4.6 that if

$$E_T = \{e \in E : e \in J \text{ for some } J \in \mathcal{T}\},$$
$$H_T = \{e \in E : e \notin J \text{ for all } J \in \mathcal{T}\},$$

then E_T and H_T are the union of (the edges of) blocks of G, necessarily disjoint. This is strengthened for postman sets in Lemmas 4.7 and 4.8, and Theorem 4.9, which gives a block decomposition of postman sets.

After studying the block structure of T-joins and postman sets, we go on to show in Theorem 5.1 that the set of edges E_O may be written as a symmetric difference of an odd number of postman sets, sharpening Corollary 3.5. Our last result, Theorem 5.2, gives further characterizations of postman sets.

In the final Section we comment on how our results are reflected on the structure of the matrix associated with the clutter $\mathcal{T}$ of minimal T-joins, and how they could be used in practice to identify those $\mathcal{T}$'s which may be looked at as families of postman sets.

3 Toida and McKee's results for postman sets

Denoting by $A \Delta Z$ the symmetric difference of the sets A and Z, we will make frequent use of the following well known result (see e.g. [1, p. 168]):

3.1 Lemma. *If J' is a T'-join, then J is a T-join if and only if $J \Delta J'$ is a $(T \Delta T')$-join.*

Since non-empty $\emptyset$-joins are disjoint unions of cycles, T-joins and cycles are inter-related:

3.2 Corollary. *If J is a non-minimal T-join, then it is the disjoint union of a minimal T-join and cycles.*

Proof. Choose $J_0 \in \mathcal{T}$ with $J_0 \subset J$ (J_0 need not be uniquely determined). Then $J \Delta J_0 = J \setminus J_0$ is a $\emptyset$-join which is the union of disjoint cycles $C_1, \dots, C_h$, with $C_i \cap J_0 = \emptyset$. Thus

$$J = J_0 \cup C_1 \cup \dots \cup C_h = J_0 \Delta C_1 \Delta \dots \Delta C_h. \qquad \square$$

Following Woodall [6], a *binary matroid* is a pair (S, W) where S is a finite set and W is a subspace of 2^S (with scalar operations modulo 2). Also, a *circuit* in a binary matroid (S, W) is a minimal non-empty set in W (identifying subsets and characteristic functions). One of the main results in McKee [2] and Woodall [6] is:

3.3 Theorem (McKee 1984, Woodall 1990). *Suppose (S, W) is a binary matroid. Then $S \in W$ if and only if each element of S lies in an odd number of circuits. Equivalently, S is the Boolean sum of some set of circuits if and only if S is the Boolean sum of the set of all circuits.*

By considering $S = E$ and W the linear subspace spanned by minimal T-joins and cycles, we have:

3.4 Corollary. *E is the symmetric difference of all postman sets and cycles, i.e. every edge belongs to an odd number of postman sets and cycles.*

Conversely, if $O \neq \emptyset$ and E is the symmetric difference of all minimal T-joins and cycles, then $T = O$.

Proof. Since E is an O-join, by Corollary 3.2 we may write E as the symmetric difference of a postman set and cycles. This implies $E = S \in W$ and Theorem 3.3 gives the first part of the result.

For the remaining part, we notice that the symmetric difference of T-joins and cycles is either a T-join or a $\emptyset$-join. If $O \neq \emptyset$, E is not a $\emptyset$-join and therefore it must be both a T-join and an O-join, i.e. $T = O$. □

Since the symmetric difference of all postman sets and cycles is either an O-join or a $\emptyset$-join depending on whether there is an odd number of postman sets, and since E is an O-join, there must be an odd number of postman sets. This is true even if $O = \emptyset$, where $\mathcal{P} = \{\emptyset\}$. Thus,

3.5 Corollary. *The family of postman sets of G has odd cardinality.*

Remark. If in Corollary 3.4 we have $O = \emptyset$, E may be written as the disjoint union of cycles, and by Theorem 3.3, E is the symmetric difference of all minimal T-joins and cycles. But we may have $T \neq O$, e.g. if G is a triangle and $|T| = 2$. When $O = \emptyset \neq T$, $|\mathcal{T}|$ must be even.

4 T-joins and Blocks

In this Section we study the connection between the block structure of the graph $G = (V, E)$ and the structure of minimal T-joins of G.

According to West [5, p. 155], a block of a loopless graph is a maximal connected subgraph with no cut-vertex. For these graphs, the only possible blocks are isolated vertices, cut-edges or maximal 2-connected subgraphs.

When loops are present, it is rather tricky to include them within blocks with this definition. As no loop is in any minimal T-join and we are only interested in edges, in this paper we will adopt the following:

4.1 Definition. A *block* of the graph G is a cut-edge, a loop, or the set of edges of a maximal 2-connected loopless subgraph of G.

We will need the following result on the intersection of two clutters:

4.2 Lemma. *Let $\mathcal{Y}$ and $\mathcal{Z}$ be clutters on the same base set X, and suppose $Y \in \mathcal{Y}$ is such that for every $Z \in \mathcal{Z}$ there exist $Y' \in \mathcal{Y}$ and $Z' \in \mathcal{Z}$ with $Y' \cap Z' = \emptyset$ and $Y' \cup Z' \subset Y \,\Delta\, Z$. Then*

$$Y \cap Z = \emptyset \quad \textit{for all } Z \in \mathcal{Z}.$$

Proof. Suppose there exists $Z \in \mathcal{Z}$ such that $Y \cap Z \neq \emptyset$ and consider $Z_0 \in \mathcal{Z}$ such that $Y \cap Z_0 \neq \emptyset$ and

$$|Y \cup Z_0| = \min\{|Y \cup Z| : Z \in \mathcal{Z} \text{ and } Y \cap Z \neq \emptyset\}.$$

By hypothesis, there exist $Y' \in \mathcal{Y}$ and $Z' \in \mathcal{Z}$ such that Y' and Z' are disjoint and $Y' \cup Z' \subset Y \,\Delta\, Z_0$.

Since $Z' \subset Y \,\Delta\, Z_0$ but $Y \cap Z_0 \neq \emptyset$, there are elements in Z_0 which are not in Z', i.e. $Z' \neq Z_0$ and since $\mathcal{Z}$ is a clutter, we must have $Z' \not\subset Z_0$. Therefore, there exist elements of Z' in Y, that is,

$$Y \cap Z' \neq \emptyset. \tag{4.1}$$

In addition, from $Z' \subset Y \,\Delta\, Z_0 \subset Y \cup Z_0$, we have $Y \cup Z' \subset Y \cup Z_0$. But since $Z' \in \mathcal{Z}$ and $Y \cap Z' \neq \emptyset$, by our choice of Z_0 we must have $|Y \cup Z'| = |Y \cup Z_0|$, and (given the inclusion),

$$Y \cup Z' = Y \cup Z_0.$$

Furthermore, $Y' \subset Y$: $Y' \setminus Y \subset (Y \cup Z_0) \setminus Y = (Y \cup Z') \setminus Y = Z' \setminus Y$, which implies $Y' \setminus Y = \emptyset$ since Z' and Y' are disjoint.

As Y and Y' are members of the clutter $\mathcal{Y}$, from $Y' \subset Y$ we have $Y' = Y$, but then $Y \cap Z' = Y' \cap Z' = \emptyset$, contradicting (4.1). □

A first consequence of Lemma 4.2 is the following:

4.3 Lemma. *Suppose $\mathcal{T} \neq \emptyset$ and let $e \in E$ be such that $e \notin J$ for all minimal T-join J, i.e. $e \in H_T$. Then*

$$C \subset H_T$$

for every cycle C with $e \in C$.

Proof. For fixed $J \in \mathcal{T}$, we use Lemma 4.2 with

$$\mathcal{Y} = \mathcal{T}, \quad \mathcal{Z} = \{C \in \mathcal{C} : e \in C\}, \quad \text{and} \quad Y = J.$$

We need to show that given $C \in \mathcal{Z}$, there exist $J' \in \mathcal{T}$ and $C' \in \mathcal{Z}$ such that $J' \cap C' = \emptyset$ and $J' \cup C' \subset J \,\Delta\, C$.

But $J_0 = J \,\Delta\, C$ is a T-join, which cannot be minimal since by hypothesis e is not in any minimal T-join and $e \in C \setminus J$. Therefore, there exists $J' \in \mathcal{T}$ such that $J' \subset J_0$ and moreover, there exists a cycle C' with $e \in C' \subset J_0 \,\Delta\, J' = J_0 \setminus J'$. □

Lemma 4.3 considers 2-connected blocks of G: either for any edge e in such a block there exists $J \in \mathcal{T}$ with $e \in J$, or else the edges of the block do not intersect any minimal T-join. Since loops are in no minimal T-join, the other interesting blocks to us are the cut-edges (bridges), and these are taken care of by Lemma 4.5. To prove it, we will use the following well known result (see e.g. [1, p. 180]):

4.4 Lemma. *If $S \subset V$ and J is a T-join, then*

$$|S \cap T| \equiv |\delta(S) \cap J| \pmod 2,$$

where $\delta(S)$ is the set of edges having exactly one endpoint in S.

In particular, if $|S \cap T|$ is odd then $\delta(S) \cap J \neq \emptyset$.

4.5 Lemma. *Suppose $\mathcal{T} \neq \emptyset$.*

(a) If $e \in E$ is a cut-edge of G, then either $e \in J$ for all T-join J or $e \notin J$ for all $J \in \mathcal{T}$.

(b) If e is not a cut-edge, then there exists a T-join J with $e \notin J$.

Proof. Suppose e is a cut-edge such that $e \in J$ for some $J \in \mathcal{T}$, and let u and v be its endpoints. If $G_u = (V_u, E_u)$ is the connected component of $G' = (V, E \setminus \{e\})$ containing u, then $\delta(V_u) \cap J = \{e\}$, and therefore, by Lemma 4.4, we must have $|V_u \cap T|$ odd and $e \in J'$ for all T-join J'.

For the second part, if e is not a cut-edge, then there exists a cycle C with $e \in C$. If $J \in \mathcal{T}$ is such that $e \in J$, then $J \,\Delta\, C$ contains a minimal T-join J' with $e \notin J'$. □

Combining Lemmas 4.3 and 4.5 we have:

4.6 Theorem. *With the previous notations, E_T is the union of some of the blocks of G, and H_T is the union of the remaining blocks of G.*

When dealing with postman sets we can say more.

4.7 Lemma. *Let $H_O = \{e \in E : e \notin \mathcal{P}$ for all $P \in \mathcal{P}\}$. Then:*

(a) If $H_O \neq \emptyset$ then H_O is a union of cycles, and if $e \in H_O$ then every cycle containing e is contained in H_O.

(b) H_O is a $\emptyset$-join, i.e. either it is a disjoint union of cycles or $H_O = \emptyset$.

(c) For arbitrary T, either no T-join intersects H_O or else every T-join does.

Proof. If $O = \emptyset$, then $H_O = E$, and the results are obvious. So let us consider the case $O \neq \emptyset$.

(a) Assume $e \in H_O$ and consider a fixed postman set P ($P \neq \emptyset$). Then $e \in E \,\Delta\, P$, but since $E \,\Delta\, P$ is a non-empty $\emptyset$-join, it is a union of cycles and there must exist one, call it C, such that $e \in C$ and $C \cap P = \emptyset$. By Lemma 4.3 we have $C \subset H_O$.

(b) Take off disjoint cycles $C_1, C_2, \ldots$ from H_O, one at a time, until there are no more cycles inside $H_O \setminus \cup_i C_i$, and consider $E' = E \setminus \cup_i C_i$ and $G' = (V, E')$.

   Then $O(G') = O(G)$. Moreover, P is a postman set in G if and only if it is a postman set in G', since if it is a postman set in G then $P \cap H_O = \emptyset$. Therefore $H_{O(G')} = H_O \setminus \cup_i C_i$.

   If $H_{O(G')} \neq \emptyset$, then (by the previous item) $H_{O(G')}$ must be a union of cycles, but we have assumed there were no cycles left in $H_O \setminus \cup_i C_i$.

(c) Suppose there exist $J \in \mathcal{T}$ and $e \in H_O \cap J$, and let us show that every $J' \in \mathcal{T}$ must intersect H_O. If $e \in J'$, then obviously J' intersects H_O. Otherwise $J \,\Delta\, J'$ contains a cycle C such that $e \in C \subset J \,\Delta\, J'$. But C is inside H_O, and therefore J' has an edge in H_O ($C \not\subset J$ since J contains no cycles). □

4.8 Lemma. *$e \in E$ is a cut-edge if and only if $e \in P$ for all $P \in \mathcal{P}$.*

Proof. Suppose e is a cut-edge. Then, since H_O is a union of cycles (Lemma 4.7), $e \notin H_O$. This implies, by the first part of Lemma 4.5, that $e \in P$ for every $P \in \mathcal{P}$.

The converse is covered by the second part of Lemma 4.5. □

Let $B_1, B_2, \ldots, B_r$ be (the edges of) the blocks of $G = (V, E)$, and for $i = 1, \ldots, r$ let O_i be the odd degree vertices of $G_i = (V_i, B_i)$, where V_i is the set of endpoints of the edges in B_i.

Since the B_i's are pairwise disjoint, we may write $E = B_1 \Delta \cdots \Delta B_r$ and therefore $O(G) = O_1 \Delta \cdots \Delta O_r$. Hence, if $\mathcal{P}_i$ is the family of postman sets in G_i and for each $i = 1, \ldots, r$ we choose $P_i \in \mathcal{P}_i$,

$$P_1 \Delta \cdots \Delta P_r = P_1 \cup \cdots \cup P_r$$

will be a postman set in G since it is an O-join having no cycles (if it contained one, it would be inside a block and hence contained in one of the P_i's).

Suppose now $P \in \mathcal{P}$ and the block B_i consists of a single edge e. If e is a loop then $\mathcal{P}_i = \{\emptyset\}$ and $e \notin P$, i.e. $P_i = \emptyset = P \cap B_i$ is the only postman set in G_i. On the other hand, if e is a cut-edge in G, $P_i = \{e\}$ is the unique postman set in $\mathcal{P}_i$ by Lemma 4.8, and since $e \in P$ we have $P_i = P \cap B_i$.

If G_i is 2-connected, we see that $E \setminus P$ is a union of disjoint cycles in G ($P \neq E$, since B_i contains a cycle), so that $B_i \cap (E \setminus P) = B_i \setminus P$ is also a union of disjoint cycles in G_i or is the empty set, and therefore $B_i \cap P$ is an O_i-join which must contain a postman set $P_i \in \mathcal{P}_i$.

Taking the union (which equals the symmetric difference) of all P_i's so constructed, we obtain a postman set in G which is contained in P and therefore must be precisely P. Thus $P_i = P \cap B_i$ for all $i = 1, \ldots, r$.

We sum up these findings in a Theorem:

4.9 Theorem. *With the previous notations, there is a one to one correspondence between $\mathcal{P}$ and $\mathcal{P}_1 \times \cdots \times \mathcal{P}_r$, given by*

$$P \to (P \cap B_1, \ldots, P \cap B_r) \quad \text{and} \quad (P_1, \ldots, P_r) \to P_1 \Delta \cdots \Delta P_r.$$

Consequently, if E' is the union of some of the blocks of G and $G' = (V, E')$, then $P \cap E'$ is a minimal $O(G')$-join for every $P \in \mathcal{P}$.

Remark. Some of the $\mathcal{P}_i$'s may reduce to the empty set if the corresponding G_i is Eulerian. In particular, this is the case of all the blocks forming H_O.

5 T-joins and Postman Sets

We have seen in Corollary 3.4 that if $S = E$ and W is the span of postman sets and cycles, then $E \in W$. We now show that we need not consider cycles if $H_O = \emptyset$:

5.1 Theorem. *Let $E_O = \{e \in E : e \in P$ for some $P \in \mathcal{P}\}$. Then there exists $\{P_1, P_2, \ldots, P_s\} \subset \mathcal{P}$ with s odd and*

$$E_O = P_1 \Delta P_2 \Delta \cdots \Delta P_s.$$

Consequently, if $H_O \neq \emptyset$, there also exist $\{C_1, C_2, \ldots, C_t\} \subset \mathcal{C} \cap H_O$ such that

$$E = P_1 \Delta \cdots \Delta P_s \Delta C_1 \Delta \cdots C_t.$$

Proof. Let us start by considering the case where E is itself a block and $O(G) \neq \emptyset$. If E consists of a single edge, whether loop or cut-edge, the result is trivial. So let us suppose G is 2-connected (and has no loops) and fix $u \in O(G)$. Since G is 2-connected, the degree of u must be greater than or equal to 3, so there are (at least) three edges e_1, e_2, e_3 incident on u.

Consider a cycle L_1 through e_2 and e_3 (and therefore not containing e_1), a second cycle L_2 through e_1 and e_3 and let $L_3 = L_1 \,\Delta\, L_2$. Then the L_i's are different non-empty $\emptyset$-joins with $L_1 \,\Delta\, L_2 \,\Delta\, L_3 = \emptyset$.

Thus, for $j = 1, 2, 3$, $E_j = E \setminus L_j$ is an O-join and if $\tilde{G}_j = (V, E_j)$ then $O(\tilde{G}_j) = O(G)$, which implies that postman sets in $\tilde{G}_j$ are also postman sets in G.

Since $|E_j| < |E|$, we may use induction on the number of edges and assume there are postman sets $P_1^j, \ldots, P_{s_j}^j$ in $\tilde{G}_j$ such that s_j is odd and

$$E_j = P_1^j \,\Delta \cdots \Delta\, P_{s_j}^j \quad \text{for } j = 1, 2, 3.$$

As $E = E_1 \,\Delta\, E_2 \,\Delta\, E_3$, the result follows for blocks.

If G is decomposed into blocks $B_1, \ldots, B_r$, as in Theorem 4.9, then for each block B_i we may find postman sets $P_1^i, \ldots, P_{s_i}^i$ with s_i odd such that

$$B_i = P_1^i \,\Delta \cdots \Delta\, P_{s_i}^i \quad \text{for } i = 1, \ldots, r,$$

and

$$E = \cup_i B_i = B_1 \,\Delta \cdots \Delta\, B_r = \mathop{\Delta}_{i,k} P_k^i, \tag{5.1}$$

where the last symmetric difference is taking over $1 \leq i \leq r$ and $1 \leq k \leq s_i$.

Selecting an index k_i, $1 \leq k_i \leq s_i$, for each $i = 1, \ldots, r$, produces a postman set

$$P_{\mathbf{k}} = P_{k_1}^1 \,\Delta\, P_{k_2}^2 \cdots \Delta\, P_{k_r}^r,$$

where $\mathbf{k}$ is the multi-index $(k_1, \ldots, k_r)$. Since each s_i is odd, there is an odd number of $\mathbf{k}$'s and also, from (5.1),

$$\mathop{\Delta}_{\mathbf{k}} P_{\mathbf{k}} = \mathop{\Delta}_{i,k} P_k^i = E,$$

where the first symmetric difference is taken over all $s_1 \times \cdots \times s_r$ possible $\mathbf{k}$'s.

For the last part of the Theorem, we know that $E = E_O \,\Delta\, H_O$ and H_O is the union of disjoint cycles if it is not empty. □

Remark. The previous Theorem implies that $|\mathcal{P}|$ is odd without need of McKee's or Woodall's results, by the same arguments used to prove Corollary 3.5.

Let us denote by R the symmetric difference $O \,\Delta\, T$ (which may be empty), and by $\mathcal{R}$ the corresponding clutter of minimal R-joins (which may only have the empty set). We have:

5.2 Theorem. *With the previous notations, if $\mathcal{T} \neq \emptyset$ and $G_T = (V, E_T)$, then the following conditions are equivalent:*

(i) T is the set of odd degree vertices of (V, E_T), i.e. $\mathcal{T}$ is the set of postman sets of G_T.

(ii) $E_T = J_1 \,\Delta\, J_2 \,\Delta \cdots \Delta\, J_s$, for some $\{J_1, J_2, \ldots, J_s\} \subset \mathcal{T}$ and odd s.

(iii) $|\mathcal{T}|$ *is odd, and* $E_T = J_1 \,\Delta\, J_2 \,\Delta \cdots \Delta\, J_s$ *for some* $\{J_1, J_2, \ldots, J_s\} \subset \mathcal{T}$.

(iv) For every $P \in \mathcal{P}$ *there exists* $J \in \mathcal{T}$ *such that* $J \subset P$.

(v) For every $P \in \mathcal{P}$ *there exist* $J_P \in \mathcal{T}$ *and* $D_P \in \mathcal{R}$ *such that* J_P *and* D_P *are disjoint and* $P = J_P \cup D_P$.

(vi) E *is the disjoint union of* E_T, E_R *and* H_O.

(vii) E_T *and* E_R *are disjoint.*

Proof. (ii) follows from (i) by Theorem 5.1, and if (ii) holds, then E_T is a T-join, which implies $O(G_T) = T$ and (i). Thus, (i) and (ii) are equivalent.

By Corollary 3.5, (iii) follows from (i) and (ii). Conversely, (iii) implies (i): by Theorem 3.3, E_T is the symmetric difference of all minimal T-joins and cycles in G_T, and since $|\mathcal{T}|$ is odd, E_T is both an $O(G_T)$-join and a T-join, which implies (i).

Let us now show the implications (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (i).

The block decomposition of T-joins given in Theorem 4.6 tells us that each of the edge sets $B_1, \ldots, B_r$ of the blocks of G is either contained in E_T (intersecting every T-join) or H_T (intersecting no T-join). Hence, using Theorem 4.9, we see that (i) implies (iv) by restricting any $P \in \mathcal{P}$ to the blocks forming E_T, i.e. by taking $J = P \cap E_T$.

Suppose now (iv) holds and for given $P \in \mathcal{P}$, let J be a T-join contained in P. Since P has no cycles, J is minimal and—even more—is the unique T-join contained in P, since if there were two, their symmetric difference would contain a cycle inside P. Let us denote by J_P this unique T-join.

If $R = O \,\Delta\, T$ and $\mathcal{R}$ are the minimal R-joins, we observe that $D_P = P \,\Delta\, J_P$ is an R-join contained in P, and by the argument just used for J_P, we obtain that $D_P \in \mathcal{R}$ and that D_P is the only R-join contained in P, so that (iv) implies (v).

We will show now that if (v) holds, then $J \cap D = \emptyset$ for every $J \in \mathcal{T}$ and $D \in \mathcal{R}$.[1] This follows from Lemma 4.2 by considering for fixed $J \in \mathcal{T}$,

$$\mathcal{Y} = \mathcal{T}, \quad \mathcal{Z} = \mathcal{R}, \quad \text{and} \quad Y = J,$$

and observing that given $D \in \mathcal{Z} = \mathcal{R}$, the O-join $J \,\Delta\, D$ contains a postman set P, which may be written as $J_P \cup D_P$ (by (v)), with $J_P \in \mathcal{T}$, $D_P \in \mathcal{R}$ and $J_P \cap D_P = \emptyset$.

Moreover, by Theorem 4.7(c), we know that either H_O intersects every T-join or it intersects none. Since $J_P \cap H_O = \emptyset$, we must have $J \cap H_O = \emptyset$ for all $J \in \mathcal{T}$. Since in Theorem 4.7(c) T is arbitrary, the same holds for R. So (v) implies (vi).

(vi) clearly implies (vii).

If (vii) holds then for $J \in \mathcal{T}$ and $D \in \mathcal{R}$ it must be $J \cap D = \emptyset$. Hence $P = J \,\Delta\, D = J \cup D$ is an O-join which has no cycles since the blocks forming E_T and E_R are disjoint. So $P \in \mathcal{P}$. Also from $J \subset E_T$, $D \subset E_R$ and $E_T \cap E_R = \emptyset$, we have $J = P \cap E_T$, that is, J is the restriction of P to the blocks forming E_T. So by Theorem 4.9, $O(G_T) = T$ which is (i). ◻

[1] Notice that this is actually condition (vii).

6 The Clutter Matrix associated with $\mathcal{T}$

Many of our results may be visualized via the 0-1 matrix $M(\mathcal{T})$ associated with the clutter $\mathcal{T}$, in which the rows are the characteristic functions of the minimal T-joins and the columns are indexed by the edges of G.

The zero-columns in this matrix correspond to blocks which do not intersect any minimal T-join, i.e. to the edges of H_T, and the remaining columns correspond to edges in E_T. Moreover, the columns with all ones correspond to bridges (though perhaps not all of the bridges).

Using the notations of Theorem 4.9, for $i = 1, \dots, r$, the matrix M_i associated with the clutter $\mathcal{P}_i$ with edges B_i, appears $\prod_{j,j \neq i} |\mathcal{P}_j|$ times as a submatrix of $M(\mathcal{P})$ in the columns corresponding to B_i. When $T = O(G)$, the zero-columns correspond to some 2-connected subgraphs and all loops, whereas the all-ones columns correspond to all bridges.

With the notations of Theorem 5.2, when $T = O(G_T) \neq O(G)$, the E_O columns in $M(\mathcal{P})$ are split into E_T and E_R columns and the matrix $M(\mathcal{P})$ may be considered as a cartesian product of $M(\mathcal{T})$ (considering edges E_T) and $M(\mathcal{R})$ (considering edges E_R) to which $|H_O|$ zero-columns have been added. In particular, the E_T columns of $M(\mathcal{T})$ show up $|\mathcal{R}|$ times as a sub-matrix of $M(\mathcal{P})$.

If we know that a matrix M is associated with a clutter of T-joins of a graph G, but do not know what G or T is, it is nevertheless very simple to test whether $T = O(G_T)$:

(a) check if $|\mathcal{T}|$ is odd, i.e. if there is an odd number of rows, and,

(b) in case $|\mathcal{T}| > 1$, check if E_T (the set of indices of the non-zero columns of M) can be written as a symmetric difference of minimal T-joins.

Since the symmetric difference of sets corresponds to addition modulo 2 of characteristic vectors, for (b) we may eliminate the zero-columns (and even the all-ones columns) and try to express a row of ones as a sum (mod 2) of some of the rows of the reduced matrix. This can be done quite efficiently (bounded by small powers of $|\mathcal{T}|$ and $|E_T|$) by using Gaussian elimination or matrix triangularization modulo 2.

We should add that, according to Novick and Sebö [3], a clutter may be recognized as a T-join clutter in polynomial time by considering the sixteen non-isomorphic minimally non T-join (binary) clutters. Therefore, the recognition of a matrix as coming from a clutter of postman sets can be done polynomially.

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Affiliations

Néstor E. Aguilera: Consejo Nacional de Investigaciones Científicas y Técnicas and Universidad Nacional del Litoral, Argentina.
e-mail: `aguilera@arcride.edu.ar`

Valeria A. Leoni: Universidad Nacional de Rosario and Fundación Antorchas, Argentina.
e-mail: `valeoni@fceia.unr.edu.ar`