The Shapley value for arbitrary families of coalitions

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Abstract

We address the problem of finding a suitable definition of a value similar to that of Shapley's, when the games are defined on a subfamily of coalitions with no structure. We present two frameworks: one based on the familiar efficiency, linearity and null player axioms, and the other on linearity and the behavior on unanimity games. We give several properties and examples in each case, and give necessary and sufficient conditions on the family of coalitions for the approaches to coincide.

1 Introduction

In cooperative games in characteristic function form we are given a finite set $N = \{1, 2, \dots, n\}$ of players and a real valued function v—which we will refer to as the *game*—defined on all the subsets of N, $\mathcal{P}(N)$, with $v(\emptyset) = 0$. A *value* is a mapping ϕ which assigns to each game v and $i \in N$ a real number $\phi_i(v)$.

Perhaps the most celebrated value is the one given by Shapley (1953), which has been extensively studied and generalized. For recent treatments of the subject, we refer the reader to the corresponding chapters in Aumann and Hart (2002), or the paper by Moretti and Patrone (2008).

When n is large, the exponentially large data involved make it certainly unrealistic to have a complete knowledge of v in practice. Even if n is small, some coalitions may not actually form, either because the players cannot meet or communicate with each other, or because of incompatibilities among them. Thus, the study of the so called restricted games was initiated. Some of these studies consider the way in which coalitions are formed, while others focus on the axioms and underlying structures, such as lattices, antimatroids, or convex geometries (see the book by Bilbao, 2000).

This work concerns the problem of finding a suitable definition of a value ϕ satisfying properties similar to those of Shapley's, when the underlying family of coalitions has no structure. Recent papers related to our work are those by Honda and Grabisch (2006), Lange and Grabisch (2006) and Faigle and Peis (2008). It is worth mentioning also the work by Castro et al. (2008), where the calculation of the Shapley value is based on sampling, alleviating the problem of huge data for large n.

We assume there is a family of *coalitions* $\mathcal{K}$, $\mathcal{K} \subset \mathcal{P}(N)$, with $\emptyset$ and N in $\mathcal{K}$, and consider the associated family of games defined on it, $\mathcal{V} = \{v : \mathcal{K} \to \mathbb{R}, v(\emptyset) = 0\}$. Notice that we don't ask for *superadditivity* of games, i.e., the condition $v(A \cup B) \geq v(A) + v(B)$ for all disjoint A and B is not required for v to be in $\mathcal{V}$ ($\mathcal{K}$ need not be closed under disjoint unions).

Note. From now on, ϕ will denote a value function defined on the set of games $\mathcal{V}$ of a family of coalitions $\mathcal{K}$.

Shapley (1953) introduced his value, denoted here by ϕ^S, by means of three axioms. Although a number of alternative set of axioms have been proposed, we discuss mostly those originally given by Shapley, adapting them to our setting.

The first axiom, *symmetry*, is defined by taking a permutation π on N and considering for $v \in \mathcal{V}$ the game $v \circ \pi$ (the composition), and the symmetry axiom asks that $\phi_i(v \circ \pi) = \phi_{\pi(i)}(v)$ for all π, v and i. Since in general we do not have $\pi(K) \in \mathcal{K}$ for all $K \in \mathcal{K}$, this axiom makes sense only partially in our context (that is, only for permutations leaving invariant $\mathcal{K}$). For example, if $N = \{1, 2, 3\}$ and $\mathcal{K} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, N\}$, we may say that the players 2 and 3 play a symmetrical role. But, if $\mathcal{K} = \{\emptyset, \{1\}, \{2\}, \{1, 3\}, N\}$, there are no symmetric players.

A related and much weaker concept is that of *equivalent players*, a property of the family $\mathcal{K}$ which does not depend on the value ϕ:

Definition 1.1. *The players i and j in N are* equivalent *(with respect to the family $\mathcal{K}$) if $\{K \in \mathcal{K} : i \in K\} = \{K \in \mathcal{K} : j \in K\}$.*

If there is no way to distinguish between two players, it is reasonable to ask that they obtain the same payoff:

Property 1.2 (EE). *ϕ is* egalitarian on equivalent players *if whenever i and j are equivalent players, $\phi_i(v) = \phi_j(v)$ for all games v.*

Notice that if $\mathcal{K} = \mathcal{P}(N)$, there are no players which are different and equivalent, and **EE** is trivially satisfied. We call it property rather than axiom, since it depends on the family of coalitions being considered.

Shapley's second axiom is currently called the *carrier* axiom, though he denominated it *efficiency*. Recall that a set R is a carrier of v if $v(S) = v(S \cap R)$ for all $S \subset N$, and the axiom states that $v(R) = \sum_{i \in R} \phi_i(v)$ for every carrier R of v. Although N is a carrier of any game, for an arbitrary family of coalitions $\mathcal{K}$ we may not have $S \cap R \in \mathcal{K}$ for all $S, R \in \mathcal{K}$ (the *intersection property*). So again, this axiom doesn't make sense in general.

Here we use the term efficiency following current practice:

Axiom 1.3 (E). *ϕ is* efficient *if $\sum_{i \in N} \phi_i(v) = v(N)$ for all $v \in \mathcal{V}$.*

Recall that for $\mathcal{K} = \mathcal{P}(N)$, $i \in N$ is a null player for v if $v(S) = v(S \cup \{i\})$ whenever $S \subset N$ and $i \notin S$, and that the carrier axiom is equivalent to **E** and the *null player axiom*. In general we may not have S and $S \cup \{i\}$ in $\mathcal{K}$, but this time we may adapt the definition.

Let $G_{\mathcal{K}} = (\mathcal{K}, \mathcal{A})$ be the directed graph whose nodes are the coalitions in $\mathcal{K}$ and (K, K') is an arc of $G_{\mathcal{K}}$ if $K \subsetneq K'$ and there is no $K'' \in \mathcal{K}$ such that $K \subsetneq K'' \subsetneq K'$. Formally, $G_{\mathcal{K}}$ is the transitive reduction of the graph induced by the inclusion on $\mathcal{K}$. For $i \in N$, let $\mathcal{A}_i = \{(K, K') \in \mathcal{A} : i \in K' \setminus K\}$.

The following definition coincides with the usual one when $\mathcal{K} = \mathcal{P}(N)$:

Definition 1.4. *$i \in N$ is a* null player *for $v \in \mathcal{V}$ if $v(K) = v(K')$ for all $(K, K') \in \mathcal{A}_i$.*

Axiom 1.5 (**N**)**.** *ϕ satisfies the* null player *axiom if for every $v \in \mathcal{V}$, $\phi_i(v) = 0$ for every null player i of v.*

The third and final axiom by Shapley, which he called *aggregation*, asks for the equality $\phi(v + w) = \phi(v) + \phi(w)$ for all $v, w \in \mathcal{V}$, but it is more convenient to ask for linearity, i.e., adding the condition $\phi(\alpha v) = \alpha \phi(v)$ for any constant $\alpha > 0$, so that we have independence from scale. Let us observe that, since we are not assuming superadditivity of games, if $v \in \mathcal{V}$ then $-v \in \mathcal{V}$, and, actually, $\mathcal{V}$ is a linear space on $\mathbb{R}$ of dimension $|\mathcal{K}| - 1$.

Axiom 1.6 (**L**)**.** *ϕ is* linear *if $\phi(\alpha v + \beta w) = \alpha \phi(v) + \beta \phi(w)$ whenever $v, w \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{R}$.*

The linearity of a value immediately leads to the study of its behavior on linear bases of $\mathcal{V}$. Usually two such bases are considered, both parameterized by $\mathcal{K}^* = \{R \in \mathcal{K} : R \neq \emptyset\}$. The first one consists of the *elementary* games E_R, defined for $R \in \mathcal{K}^*$ by

$$E_R(K) = \begin{cases} 1 & \text{if } K = R, \\ 0 & \text{otherwise,} \end{cases}$$

and the other consists of the *unanimity* games U_R ($R \in \mathcal{K}^*$), defined by

$$U_R(K) = \begin{cases} 1 & \text{if } R \subset K, \\ 0 & \text{otherwise.} \end{cases}$$

Unlike elementary games, unanimity games are superadditive and therefore better suited to cooperative game theory. On the other hand, elementary games are easier to work with because of the simple representation

$$v = \sum_{K \in \mathcal{K}^*} v(K)\, E_K \quad \text{for all } v \in \mathcal{V}. \tag{1.1}$$

Given that

$$U_R = \sum_{T : R \subset T} E_T, \tag{1.2}$$

it is easy to express unanimity games in terms of elementary games. The opposite representation—of elementary games in terms of unanimity games—is often called the *Möbius transform*, which for $\mathcal{K} = \mathcal{P}(N)$ takes the simple form

$$E_T = \sum_{R : T \subset R} (-1)^{|R|-|T|}\, U_R.$$

Assuming a linear order on $\mathcal{K}^*$ compatible with the inclusion of coalitions, (1.2) may be represented by an upper triangular binary matrix having ones in the diagonal, and therefore its inverse is an upper triangular integer matrix having ones in the diagonal. However, the entries of this inverse matrix need not be 0 or ± 1 as is the case for $\mathcal{K} = \mathcal{P}(N)$. For instance, if $\mathcal{K} = \{\emptyset, K_1 = \{1\}, K_2 = \{1,2,3\}, K_3 = \{1,2,4\}, K_4 = \{1,3,4\}, N = \{1,2,3,4\}\}$, then $E_{K_1} = U_{K_1} - U_{K_2} - U_{K_3} - U_{K_4} + 2\, U_N$.

Shapley showed that, when $\mathcal{K} = \mathcal{P}(N)$, the carrier and symmetry axioms imply the following property, which we take as axiom (cf. Aumann, 1990):

Axiom 1.7 (**EU**). *ϕ is* egalitarian on unanimity games *if for all $T \in \mathcal{K}^*$,*

$$\phi_i(U_T) = \begin{cases} 1/|T| & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that neither the carrier nor the symmetry axioms can be used for general families, but **EU** makes perfect sense in our setting.

In his 1953 paper, Shapley presented also a bargaining model in which, starting with a single player, players are added one at a time until everyone has been admitted, and each player is given her marginal contribution (according to the game v) when incorporated. If every order is possible and all orders are equally probable, then $\phi_i^S(v)$ is the corresponding expectation.

These marginal contributions are quite apparent in the usual expression for ϕ^S,

$$\phi_i^S(v) = \sum_{s=1}^{n} \frac{(s-1)!\,(n-s)!}{n!} \sum_{i \in S : |S| = s} (v(S) - v(S \setminus \{i\})), \tag{1.3}$$

which leads to the following definition:

Definition 1.8. *ϕ is* marginalist *if there exist coefficients $\lambda_i(K, K') \in \mathbb{R}$, defined for each $i \in N$ and $(K, K') \in \mathcal{A}_i$, such that*

$$\phi_i(v) = \sum_{(K,K') \in \mathcal{A}_i} \lambda_i(K, K')\, (v(K') - v(K)). \tag{1.4}$$

When convenient, we set $\lambda_i(K, K') = 0$ if $i \notin K' \setminus K$, so that the sum in (1.4) may be taken over all $(K, K') \in \mathcal{A}$, and not just $\mathcal{A}_i$.

If $\mathcal{K} = \mathcal{P}(N)$, there is only one player in $K' \setminus K$ if $(K, K') \in \mathcal{A}$, and precisely $\mathcal{K}$ is *regular* (Honda and Grabisch, 2006) if $|K' \setminus K| = 1$ for all $(K, K') \in \mathcal{A}$. In general, we may have several players in $K' \setminus K$, and it is reasonable to distribute the marginal gains equally among the members of the arcs.

Property 1.9 (**IS**)**.** *A marginalist value* ϕ *has* internal symmetry *if it may represented in the form* (1.4) *with*

$$\lambda_i(K, K') = \lambda_j(K, K') \quad \text{for all } (K, K') \in \mathcal{A} \text{ and } i, j \in K' \setminus K. \tag{1.5}$$

As is the case of **EE**, **IS** is satisfied by every value when $\mathcal{K} = \mathcal{P}(N)$. Actually, **IS** implies **EE**, since i and j are equivalent players if and only if $\mathcal{A}_i = \mathcal{A}_j$. As the representation as a marginalist value need not be unique, it may be the case that in one representation (1.5) is satisfied but not in another (see Example 2.4).

Summary of results

In this paper we present two different frameworks for defining a generalized Shapley value.

In Section 2 we discuss the first one, based on **N**, **L** and **E**, and related to Shapley's bargaining model. We show that **N** and **L** are equivalent to marginalism (Theorem 2.3), though the representation may not be unique (Example 2.4), and we characterize those $\mathcal{K}$ for which it is (Theorem 2.5). By adding **E**, we show that a marginalist value determines a flow in $G_{\mathcal{K}}$ (Theorem 2.6), and by adding **IS**, we obtain the converse (Theorem 2.7).

N, **L** and **E** are not enough to uniquely define ϕ^S when $\mathcal{K} = \mathcal{P}(N)$, and another axiom, such as symmetry, is needed. This remains true for general $\mathcal{K}$, even if **IS** is required, and in the last part of Section 2 we discuss two different approaches, both based on maximal chains. The first one, given in Section 2.1, depends on the decomposition of a flow along $\emptyset$–N paths in $G_{\mathcal{K}}$ (Theorem 2.9 and Example 2.10), and is closely related to the probabilistic study by Weber (1988). We present different models according to the weights given to the paths, and show their differences in Example 2.11. The second approach is a generalization of the potential flow model of Lange and Grabisch (2006), and we study it in Section 2.2, obtaining uniqueness (Theorem 2.17) when the *regularity axiom*, **R**, is satisfied.

In Section 3 we study the second framework, which is inspired in the Hart and Mas-Colell (1989) treatment of the potential. We use **L** and **EU** to find existence and uniqueness of a value, ϕ^U, which also satisfies **E** and **EE** (Theorem 3.1). We devote the remainder of this section to study when ϕ^U is marginalist, giving first some examples and then some necessary conditions under sensible hypotheses. Lemma 3.3 reveals a quantity related to the coefficients of ϕ^S in (1.3) and the potential models, and Theorem 3.5 exposes a property, P-1, on coalitions. As a consequence, we obtain that for regular families, ϕ^U is marginalist if and only if $\mathcal{K} = \mathcal{P}(N)$ (Corollary 3.6). Finally, Theorem 3.7 shows that P-1 implies that ϕ^U is marginalist, and in Example 3.8 we show that P-1 may not be simplified easily.

In the last section we make a few final comments.

2 Marginalist and efficient values

For $i \in N$, let us define

$$\begin{aligned}\mathcal{K}_i^+ &= \{K \in \mathcal{K} : (K, K') \in \mathcal{A}_i \text{ for some } K' \in \mathcal{K}\},\\ \mathcal{K}_i^- &= \{K \in \mathcal{K} : (K', K) \in \mathcal{A}_i \text{ for some } K' \in \mathcal{K}\},\\ \mathcal{K}_i &= \mathcal{K}_i^+ \cup \mathcal{K}_i^-,\end{aligned}$$

so that $\mathcal{K}_i$ is the set of coalitions which are endpoints of an arc in $\mathcal{A}_i$. Let us denote by $\mathcal{C}_{i,k} = (\mathcal{K}_{i,k}, \mathcal{A}_{i,k})$, $k = 1, \ldots, c(i)$, the (weak) components induced by $\mathcal{A}_i$. Notice that $\mathcal{C}_{i,k}$ is directed.

Note. If a graph G is directed, we will denote by G° the underlying undirected version.

The following results are easy to prove.

Lemma 2.1. *ϕ satisfies **N** if and only if $\phi_i(v) = 0$ whenever $v \in \mathcal{V}$ and $i \in N$ are such that for all $k = 1, \ldots, c(i)$ and $(R, R') \in \mathcal{A}_{i,k}$ there holds $v(R) = v(R')$.*

Lemma 2.2. *If ϕ satisfies **N**, $i \in N$, and $S \notin \mathcal{K}_i$, then $\phi_i(E_S) = 0$.*

It is clear that a marginalist value satisfies **N** and **L**, and we now show that there are no others (cf. Bilbao, 2000, Ch. 7):

Theorem 2.3. *ϕ is marginalist if and only if it satisfies **N** and **L**.*

Proof. If ϕ satisfies **N** and **L**, using (1.1) and Lemma 2.2 we obtain

$$\phi_i(v) = \sum_{K \in \mathcal{K}_i} v(K)\, \phi_i(E_K) = \sum_{k=1}^{c(i)} \sum_{K \in \mathcal{K}_{i,k}} v(K)\, \phi_i(E_K). \tag{2.1}$$

Letting

$$v_{i,k} = \sum_{K \in \mathcal{K}_{i,k}} E_K \quad \text{for } k = 1, \ldots, c(i), \tag{2.2}$$

we have

$$v_{i,k}(R) = v_{i,k}(R') \in \{0, 1\} \quad \text{for } (R, R') \in \mathcal{A}_i,\ k = 1, \ldots, c(i),$$

since either R and R' are in the same component $\mathcal{C}_{i,k}$ and exactly one term in (2.2) is 1, or else they are in different components and all terms are 0. Hence, using **N**,

$$\phi_i(v_{i,k}) = 0 \quad \text{for } k = 1, \ldots, c(i). \tag{2.3}$$

For each component $\mathcal{C}_{i,k}$, let us fix $K_{i,k} \in \mathcal{K}_{i,k}$. From (2.2) we may write

$$E_{K_{i,k}} = v_{i,k} - \sum_{K \in \mathcal{K}_{i,k} : K \neq K_{i,k}} E_K,$$

and using **L** and (2.3),

$$\phi_i(E_{K_{i,k}}) = \phi_i(v_{i,k}) - \sum_{K\in\mathcal{K}_{i,k}:K\neq K_{i,k}} \phi_i(E_K) = - \sum_{K\in\mathcal{K}_{i,k}:K\neq K_{i,k}} \phi_i(E_K). \quad (2.4)$$

Thus, using (2.1) and (2.4),

$$\begin{aligned}\phi_i(v) &= \sum_{k=1}^{c(i)} \sum_{K\in\mathcal{K}_{i,k}} v(K)\,\phi_i(E_K) \\ &= \sum_{k=1}^{c(i)} \Big(v(K_{i,k})\,\phi_i(E_{K_{i,k}}) + \sum_{K\in\mathcal{K}_{i,k}:K\neq K_{i,k}} v(K)\,\phi_i(E_K)\Big) \\ &= \sum_{k=1}^{c(i)} \sum_{K\in\mathcal{K}_{i,k}:K\neq K_{i,k}} (v(K)-v(K_{i,k}))\,\phi_i(E_K).\end{aligned} \quad (2.5)$$

For each $K\in\mathcal{K}_{i,k}$, $K\neq K_{i,k}$, let $(K_0 = K_{i,k}, K_1, \ldots, K_s = K)$ be a path joining $K_{i,k}$ and K in $\mathcal{C}^\circ_{i,k}$, and let $P_{i,k}(K)$ be the subgraph of $\mathcal{C}_{i,k}$ induced by this path. We write each term of the sum in (2.5) as a telescoping sum:

$$\begin{aligned}(v(K)-v(K_{i,k}))\,\phi_i(E_K) &= \sum_{j=1}^{s} (v(K_j)-v(K_{j-1}))\,\phi_i(E_K) \\ &= \sum_{(K',K'')\in\mathcal{A}_{i,k}} \lambda_{i,k,K}(K',K'')\,(v(K'')-v(K')),\end{aligned}$$

where for $K\neq K_{i,k}$,

$$\lambda_{i,k,K}(K',K'') = \begin{cases} \phi_i(E_K) & \text{if } (K',K'') \text{ is an arc in } P_{i,k}(K), \\ -\phi_i(E_K) & \text{if } (K'',K') \text{ is an arc in } P_{i,k}(K), \\ 0 & \text{otherwise.}\end{cases}$$

and $\lambda_{i,k,K_{i,k}}(K',K'') = 0$ for all $(K',K'')\in\mathcal{A}_i$.

We may rewrite (2.5) as

$$\begin{aligned}\phi_i(v) &= \sum_{k=1}^{c(i)} \sum_{K\in\mathcal{K}_{i,k}} \sum_{(K',K'')\in\mathcal{A}_{i,k}} \lambda_{i,k,K}(K',K'')\,(v(K'')-v(K')) \\ &= \sum_{(K',K'')\in\mathcal{A}} \lambda_i(K',K'')\,(v(K'')-v(K')),\end{aligned}$$

where

$$\lambda_i(K',K'') = \sum_{k=1}^{c(i)} \sum_{K\in\mathcal{K}_{i,k}} \lambda_{i,k,K}(K',K'') \quad \text{if } (K,K')\in\mathcal{A}_i,$$

and $\lambda_i(K',K'') = 0$ otherwise. □

Since the path in the previous proof need not be unique, it comes as no surprise that there may be different representations in the form (1.4).

Example 2.4. Let $\mathcal{K} = \{\emptyset, K_1 = \{1\}, K_2 = \{2\}, K_3 = \{1,2,3,4\}, K_4 = \{1,2,3,5\}, N = \{1,2,3,4,5\}\}$. The value ϕ defined by

$$\phi_i(v) = \begin{cases} v(K_1) & \text{if } i = 1, \\ \dfrac{1}{3}\,(v(K_3) - v(K_1)) & \text{if } i = 2, 3, 4, \\ v(N) - v(K_3) & \text{if } i = 5, \end{cases}$$

may be written also as

$$\phi_1(v) = v(K_1), \qquad \phi_2(v) = \phi_4(v) = \frac{1}{3}\,(v(K_3) - v(K_1)),$$
$$\phi_3(v) = \frac{1}{3}\,(v(K_3) - v(K_2)) - \frac{1}{3}\,(v(K_4) - v(K_2)) + \frac{1}{3}\,(v(K_4) - v(K_1)),$$
$$\phi_5(v) = v(N) - v(K_3).$$

In one representation ϕ satisfies (1.5) (and therefore ϕ satisfies **IS**), but not in the other. Moreover, ϕ is strongly monotone in the sense of Young (1985), although in the second representation some of the coefficients are negative (see also Remark 2.8). ◇

Theorem 2.5. *If $\mathcal{A}_i^\circ$ is acyclic for all $i \in N$, then the representation of the marginalist value ϕ in (1.4) is unique.*

Conversely, if there exists a value ϕ whose representation in the form (1.4) is unique, then $\mathcal{A}_i^\circ$ is acyclic for all $i \in N$.

Proof. If $\mathcal{A}_i^\circ$ is acyclic for all $i \in N$, by taking differences between two representations we may assume that there exist coefficients $\mu_i(K, K')$, defined for $i \in N$ and $(K, K') \in \mathcal{A}$, such that

$$\sum_{(K,K') \in \mathcal{A}_i} \mu_i(K, K')\,(v(K') - v(K)) = 0 \quad \text{for all } v \in \mathcal{V}.$$

By taking $v = E_K$, we obtain

$$\sum_{(K,K') \in \mathcal{A}_i} \mu_i(K, K') = 0 \qquad \text{for all } K \in \mathcal{K}_i^+,$$
$$\sum_{(K',K) \in \mathcal{A}_i} \mu_i(K, K') = 0 \qquad \text{for all } K \in \mathcal{K}_i^-.$$

If K is a leaf of the tree $\mathcal{C}_{i,k}^\circ$ and (K, K') is the only edge of $\mathcal{C}_{i,k}^\circ$, we must have $\mu_i(K, K') = 0$ or $\mu_i(K', K) = 0$, depending on whether K is in $\mathcal{K}_i^+$ or $\mathcal{K}_i^-$. Eliminating K and this edge, repeating the procedure on each remaining leaf and then on every component, we see that necessarily $\mu_i(K, K') = 0$ for all $(K, K') \in \mathcal{A}_i$.

Conversely, if for some $i \in N$ a component $\mathcal{C}_{i,k}$ contains an undirected cycle $\sigma = (K_0, K_1, \ldots, K_\ell = K_0)$, we may assume $K_0 \in \mathcal{K}_i^+$ and we may write, since ℓ is necessarily even,

$$0 = \sum_{s=1}^{\ell} (-1)^s (v(K_s) - v(K_{s-1})).$$

Hence, the null value ($\phi \equiv 0$) has at least two different representations, and therefore so does any marginalist value. ☐

Our next result relates marginalist and efficient values with flows in $G_{\mathcal{K}}$ (cf. Bilbao, 2000, Ch. 7).

Setting $\mathcal{K}^{**} = \{K \in \mathcal{K} : K \neq \emptyset, N\}$, a *flow* in $G_{\mathcal{K}}$ is a function $f : \mathcal{A} \to \mathbb{R}$ such that

$$\sum_{K' : (K', K) \in \mathcal{A}} f(K', K) = \sum_{K' : (K, K') \in \mathcal{A}} f(K, K') \quad \text{for all } K \in \mathcal{K}^{**},$$

i.e., the conservation equations hold at interior nodes. Noticing that we don't ask for $f \geq 0$, the conservation equations imply

$$\sum_{K : (\emptyset, K) \in \mathcal{A}} f(\emptyset, K) = \sum_{K : (K, N) \in \mathcal{A}} f(K, N),$$

and this common number is called the *value* of the flow (regrettably, the term "value" is used for two quite different objects).

If $\chi(A, x)$ is the indicator function of the set A (that is, $\chi(A, x) = 1$ if $x \in A$ and is 0 otherwise), and ϕ is marginalist and written in the form (1.4),

$$\begin{aligned}
\sum_{i \in N} \phi_i(v) &= \sum_{i \in N} \sum_{(K, K') \in \mathcal{A}_i} \lambda_i(K, K') \, (v(K') - v(K)) \\
&= \sum_{(K, K') \in \mathcal{A}} (v(K') - v(K)) \Big(\sum_{i \in N} \lambda_i(K, K') \, \chi(\mathcal{A}_i, (K, K')) \Big) \qquad (2.6) \\
&= \sum_{(K, K') \in \mathcal{A}} \Lambda(K, K') \, (v(K') - v(K)),
\end{aligned}$$

where

$$\Lambda(K, K') = \sum_{i \in N} \lambda_i(K, K') \, \chi(\mathcal{A}_i, (K, K')) = \sum_{i : i \in K' \setminus K} \lambda_i(K, K').$$

If $R \in \mathcal{K}^{**}$, $v = E_R$, and ϕ is efficient, from (2.6) we obtain, successively,

$$\sum_{i \in N} \phi_i(E_R) = - \sum_{(R, K') \in \mathcal{A}} \Lambda(R, K') + \sum_{(K, R) \in \mathcal{A}} \Lambda(K, R) = E_R(N) = 0,$$

$$\sum_{(R, K') \in \mathcal{A}} \Lambda(R, K') = \sum_{(K, R) \in \mathcal{A}} \Lambda(K, R).$$

On the other hand, if $v = E_N$, (2.6) turns into

$$\sum_{i \in N} \phi_i(E_N) = \sum_{(K,N) \in \mathcal{A}} \Lambda(K,N) = E_N(N) = 1.$$

Thus we have:

Theorem 2.6. *If ϕ is marginalist (written in the form (1.4)) and efficient, then the quantities*

$$\Lambda(K,K') = \sum_{i : i \in K' \setminus K} \lambda_i(K,K'), \quad (K,K') \in \mathcal{A},$$

define a flow of value 1 in $G_{\mathcal{K}}$.

We may always find a converse by forcing internal symmetry:

Theorem 2.7. *Suppose Λ is a flow of value 1 in $G_{\mathcal{K}}$. Then, if*

$$\lambda_i(K,K') = \frac{\Lambda(K,K')}{|K' \setminus K|} \qquad \text{for all } i \in K' \setminus K,$$

*the value ϕ defined by (1.4) is a marginalist and efficient value satisfying **IS**.*

Proof. Only efficiency needs to be checked, and this is easily done. □

Remark 2.8. In our setting, the strong monotonicity of Young (1985) may be expressed as: if $v \in \mathcal{V}$ satisfies $v(K') \geq v(K)$ for all $(K,K') \in \mathcal{A}_i$, then $\phi_i(v) \geq 0$. Lange and Grabisch (2006) show that for marginalist values on regular families, it is equivalent to having nonnegative coefficients λ in (1.4). This does not hold in general, as it is possible to construct a marginalist value satisfying **E**, **IS**, strong monotonicity, unique representation, but some of the coefficients in (1.4) are negative (see also Example 2.4). ♢

2.1 Maximal chains

Every flow in $G_{\mathcal{K}}$ may be decomposed (perhaps not uniquely) along paths from $\emptyset$ to N, which are precisely the maximal chains for inclusion. Let us suppose, then, that $W = (K_0, K_1, \ldots, K_\ell)$, with $K_0 = \emptyset$ and $K_\ell = N$, is a maximal chain for the inclusion in $\mathcal{K}$, and define the value ψ_W by

$$\psi_{W,i}(v) = \sum_{j=1}^{\ell(W)} \frac{v(K_j) - v(K_{j-1})}{|K_j| - |K_{j-1}|} \chi(K_j \setminus K_{j-1}, i) \quad \text{for } i \in N.$$

ψ_W is a marginalist and efficient value satisfying **IS**. Thus, denoting by $\mathcal{W}$ the family of all maximal chains, if $\gamma : \mathcal{W} \to \mathbb{R}$ satisfies

$$\sum_{W \in \mathcal{W}} \gamma(W) = 1, \tag{2.7}$$

then

$$\phi(v) = \sum_{W \in \mathcal{W}} \gamma(W)\, \psi_W(v), \tag{2.8}$$

is also marginalist, efficient and satisfies **IS**.

Conversely, if Λ is a flow of value 1 in $G_{\mathcal{K}}$, we may decompose it along $\emptyset$–N paths and find coefficients γ so that (2.7) holds and

$$\Lambda(K, K') = \sum_{W \in \mathcal{W} : (K, K') \in W} \gamma(W) \quad \text{for all } (K, K') \in \mathcal{A}, \tag{2.9}$$

which induces the value

$$\phi_i(v) = \sum_{(K, K') \in \mathcal{A}_i} \frac{\Lambda(K, K')}{|K'| - |K|} \, (v(K') - v(K)). \tag{2.10}$$

We may summarize these results as:

Theorem 2.9. *The following are equivalent:*

(i) There exists $\gamma : \mathcal{W} \to \mathbb{R}$ satisfying (2.7) such that ϕ may be written as in (2.8).

(ii) There exists $\Lambda : \mathcal{A} \to \mathbb{R}$ defining a flow in $G_{\mathcal{K}}$ of value 1 such that ϕ may be written as in (2.10).

*(iii) ϕ is marginalist, efficient and satisfies **IS**.*

The coefficients γ need not be unique or nonnegative:

Example 2.10. Let $\mathcal{K} = \{\emptyset, K_1 = \{1\}, K_2 = \{2\}, K_3 = \{1, 2\}, K_4 = \{1, 2, 3\}, K_5 = \{1, 2, 4\}, N = \{1, 2, 3, 4\}\}$. The maximal chains or $\emptyset$–N paths are

$$\begin{aligned} W_1 &= (\emptyset, K_1, K_3, K_4, N), & W_2 &= (\emptyset, K_1, K_3, K_5, N), \\ W_3 &= (\emptyset, K_2, K_3, K_4, N), & W_4 &= (\emptyset, K_2, K_3, K_5, N), \end{aligned}$$

so that if $\Lambda(K, K') = 1/2$ for all $(K, K') \in \mathcal{A}$, and ϕ is defined by (2.10), then

$$\phi = \psi_{W_1} + \psi_{W_3} = \psi_{W_2} + \psi_{W_4} = \alpha\,(\psi_{W_1} + \psi_{W_3}) + \beta\,(\psi_{W_2} + \psi_{W_4}),$$

as long as $\alpha + \beta = 1$ (so that we may take, say, $\alpha < 0$). $\diamondsuit$

If γ satisfies (2.7) and $\gamma(W) \geq 0$ for all $W \in \mathcal{W}$, we may think either that ϕ is a convex combination of the ψ_W's, or, in the spirit of Shapley's bargaining model, that the coefficient $\gamma(W)$ is the probability of the chain W being chosen. At any rate, when varying γ (keeping nonnegativity), we obtain what might be called the Weber set (Weber, 1988).

As already mentioned, **N**, **L** and **E** jointly do not guarantee uniqueness of the value (and much less of the coefficients γ), even if **IS** is required. Let us give some examples of axioms or properties we may require of the coefficients γ so as to obtain uniqueness.

1. All chains have equal weight or probability, $\gamma(W) = 1/|\mathcal{W}|$. This is the case of the Shapley value, when $\mathcal{K} = \mathcal{P}(N)$, and corresponds to an indifference principle, an idea which goes back to Laplace.

2. A chain has more weight if it is longer. For instance, if proportional to the length, we would set

$$\gamma(W) = \frac{|W|}{\sum_{W' \in \mathcal{W}} |W'|}.$$

3. Conversely, we may ask it to have less weight if it is longer. For instance, we may take the harmonic mean of the cardinals,

$$\gamma(W) = \frac{1/|W|}{\sum_{W' \in \mathcal{W}} 1/|W'|}.$$

4. Borrowing from Information Theory, we could define the *entropy* of a chain $W = (K_0 = \emptyset, K_1, \dots, K_\ell = N)$ by

$$H(W) = -\sum_{k=1}^{\ell} p_k \log_2 p_k,$$

   where $p_k = |K_k \setminus K_{k-1}|/n$, and then weigh according to this index. For instance, substituting $H(W)$ for $|W|$ in models 2 or 3.

Notice that in all these models we have $\gamma(W) \geq 0$, and that the corresponding values coincide if $|W| = n$ for all $W \in \mathcal{W}$. In particular, in all of them we get back the Shapley value, ϕ^S, if $\mathcal{K} = \mathcal{P}(N)$.

The next example illustrates the differences between these models.

Example 2.11. Consider $\mathcal{K} = \{\emptyset, K_1 = \{1\}, K_2 = \{1,2\}, K_3 = \{3\}, N = \{1,2,3\}\}$, having just two maximal chains, $W_1 = (\emptyset, K_1, K_2, N)$ and $W_2 = (\emptyset, K_3, N)$. Then:

$$\begin{aligned} &\gamma(W_1) = \gamma(W_2) = 0.5, && \text{in model 1,} \\ &\gamma(W_1) = 0.6, \quad \gamma(W_2) = 0.4, && \text{in model 2,} \\ &\gamma(W_1) = 0.4, \quad \gamma(W_2) = 0.6, && \text{in model 3,} \end{aligned}$$

and, assuming γ proportional to the entropies, in model 4 we have

$$\begin{aligned} H(W_1) &\simeq 1.58496, & H(W_2) &\simeq 0.918296, \\ \gamma(W_1) &\simeq 0.63316, & \gamma(W_2) &\simeq 0.36684. \end{aligned} \qquad \diamondsuit$$

Example 3.2 shows another possibility when $\mathcal{K}$ has a particular structure, in which we still have $\gamma \geq 0$, but is not included in any of the models above.

2.2 The potential model of Lange and Grabisch

Taking a somewhat different route from the previous subsection, we follow now Lange and Grabisch (2006), extending their potential flow model. They obtained uniqueness of the value for regular families by adding the following:

Axiom 2.12 (**R**)**.** *A value $\phi : \mathcal{V} \to \mathbb{R}^n$ represented in the form* (2.10) *satisfies the* regularity axiom *if*

$$\sum_{(K,K')\in W} \Lambda(K,K') \tag{2.11}$$

is constant for all $W \in \mathcal{W}$, i.e, it is independent of $W \in \mathcal{W}$.

Recall that a flow $f : \mathcal{A} \to \mathbb{R}$ is *potential* if there exists $P : \mathcal{K} \to \mathbb{R}$, the *potential of the flow*, such that (in our setting)

$$f(K,K') = P(K') - P(K) \quad \text{for all } (K,K') \in \mathcal{A}. \tag{2.12}$$

The potential P is not unique, since $P + c$ is also a potential for any constant c, so it usual to fix the potential at a point, for instance, by asking $P(\emptyset) = 0$.

For $K \in \mathcal{K}$, let us set

$$\begin{aligned} \delta^+(K) &= \{K' \in \mathcal{K} : (K,K') \in \mathcal{A}\}, \\ \delta^-(K) &= \{K' \in \mathcal{K} : (K',K) \in \mathcal{A}\}, \\ \delta(K) &= \delta^+(K) \cup \delta^-(K), \end{aligned}$$

that is, $\delta(K)$ is the set of neighbors of K in $G^\circ_{\mathcal{K}}$, and let

$$\mathrm{d}^-(K) = |\delta^-(K)|, \quad \mathrm{d}^+(K) = |\delta^+(K)|, \quad \mathrm{d}(K) = |\delta(K)|,$$

be the corresponding degrees.

Lemma 2.13. *If f is a flow in $G_{\mathcal{K}}$ with potential P, then*

$$P(K) = \frac{1}{\mathrm{d}(K)} \sum_{K'\in\delta(K)} P(K') \quad \textit{for all } K \neq \emptyset, N. \tag{2.13}$$

Proof. Since f is a flow, $\sum_{(K',K)\in\mathcal{A}} f(K',K) = \sum_{(K,K')\in\mathcal{A}} f(K,K')$ for all $K \in \mathcal{K}^{**}$, or, using (2.12),

$$\sum_{(K',K)\in\mathcal{A}} (P(K) - P(K')) = \sum_{(K,K')\in\mathcal{A}} (P(K') - P(K)) \quad \text{for all } K \in \mathcal{K}^{**}.$$

In turn, these equations are equivalent to

$$\mathrm{d}(K)\, P(K) - \sum_{K'\in\delta(K)} P(K') = 0 \quad \text{for all } K \in \mathcal{K}^{**}. \tag{2.14}$$

□

Lemma 2.14. *Given a family $\mathcal{K}$ and $\alpha \in \mathbb{R}$, there exists a unique potential flow whose potential P satisfies $P(\emptyset) = 0$ and $P(N) = \alpha$.*

Proof. The equations (2.14) and the conditions $P(\emptyset) = 0$ and $P(N) = \alpha$, determine a linear system of the form

$$(D - A)P = b, \tag{2.15}$$

where $A \in \mathbb{R}^{\mathcal{K}^{**}}$ is the adjacency matrix of the graph $\widehat{G}$ induced by $\mathcal{K}^{**}$ in $G^{\circ}_{\mathcal{K}}$, and D is a diagonal matrix with $D_{KK} = \mathrm{d}(K)$.

$D - A$ is diagonally dominant, since

$$\sum_{K' \in \mathcal{K}^{**}} A_{KK'} = |\delta(K) \cap \mathcal{K}^{**}| \leq \mathrm{d}(K) \quad \text{for all } K \in \mathcal{K}^{**},$$

with strict inequality if $\emptyset \in \delta(K)$ or $N \in \delta(K)$. Hence, if $\widehat{G}$ is connected, the matrix $A - D$ is irreducible and therefore invertible (see, e.g., Ortega, 1990), and (2.15) has a unique solution P. If $\widehat{G}$ is not connected (as in Example 2.11), we repeat this procedure for each component. The equations (2.14), equivalent to those in (2.13), ensure that (2.12) defines a flow. □

By choosing a suitable constant α and using linearity, we obtain:

Corollary 2.15. *Given a family $\mathcal{K}$, there exists a unique potential flow of value* 1 *(and $P(\emptyset) = 0$).*

Lemma 2.16. *Let ϕ be representable in the form* (2.10)*. Then, ϕ satisfies* ***R*** *if and only if Λ is a potential flow, i.e., there exists $P : \mathcal{K} \to \mathbb{R}$ such that $P(\emptyset) = 0$ and*

$$\Lambda(K, K') = P(K') - P(K) \quad \textit{for all } (K, K') \in \mathcal{A}. \tag{2.16}$$

Proof. Assume ϕ is in the form (2.10) and satisfies **R**. For $K \in \mathcal{K}$, let us consider a chain W such that $K \in W$, and define

$$P(K) = \sum_{\substack{(K', K'') \in W \\ K'' \subset K}} \Lambda(K', K'').$$

$P(K)$ does not depend on the chosen chain, since if W' is another chain with $K \in W'$, and we consider the chain W'' which coincides with W' up to K and then continues as in W, using **R** we would have

$$\sum_{(K', K'') \in W} \Lambda(K', K'') = \sum_{(K', K'') \in W'} \Lambda(K', K'') = \sum_{(K', K'') \in W''} \Lambda(K', K''),$$

and since W and W'' coincide from K on, and W' and W'' up to K,

$$\sum_{\substack{(K', K'') \in W \\ K'' \subset K}} \Lambda(K', K'') = \sum_{\substack{(K', K'') \in W'' \\ K'' \subset K}} \Lambda(K', K'') = \sum_{\substack{(K', K'') \in W' \\ K'' \subset K}} \Lambda(K', K'').$$

So, if $(K, K') \in \mathcal{A}$, we may choose a chain containing K and K', obtaining $P(K') = P(K) + \Lambda(K, K')$.

Conversely, if ϕ may be written in the form (2.10) and there exists a function P such that (2.16) holds, the sum in (2.11) is telescopic and reduces to $P(N) - P(\emptyset) = P(N)$ for every $W \in \mathcal{W}$. □

Theorem 2.9, Corollary 2.15 and Lemma 2.16 imply:

Theorem 2.17. *Give a family $\mathcal{K}$, there exists a unique ϕ satisfying* ***N****,* ***L****,* ***E****,* ***IS*** *and* ***R****. Moreover, ϕ may be written in the form (2.10), with Λ defined by (2.16), where P is a potential satisfying $P(\emptyset) = 0$ and $P(N) = 1$.*

Let us make a few comments on this model.

- Although there are some similarities between this model and that of Hart and Mas-Colell (1989), the latter is a potential on the value for a given game, and the former is a potential on the coefficients (the flow).
- Lange and Grabisch give an example of their model on regular families in which some of the coefficients Λ are negative, and hence some of the coefficients γ (by (2.9)), unlike the models we presented in Section 2.1. Of course, the corresponding value in their example is not strongly monotone in the sense of Young, 1985 (see Remark 2.8).
- Furthermore, the potential model applied to the family $\mathcal{K}$ in Example 2.10, yields $\Lambda(K, K') = 1/2$ for all $(K, K') \in \mathcal{A}$. Therefore, the coefficients γ of Section 2.1 are not uniquely defined in general.
- Comparing with the models in Section 2.1, in the potential flow model some arcs and chains with fewer information could receive a larger weight. For instance, when applied to Example 2.11, it coincides with the harmonic means model 3.

3 Unanimity games

The following result is immediate, and we omit its proof:

Theorem 3.1. *There exists a unique value, ϕ^U, satisfying* ***L*** *and* ***EU****. Moreover, ϕ^U satisfies* ***E*** *and* ***EE****.*

Notice that:

- ϕ^U does not depend on the definition of $G_{\mathcal{K}}$. It is well defined as long as there is a nonempty coalition in the family, and $N \in \mathcal{K}$ is not needed.
- ϕ^U is the restriction of the Shapley value ϕ^S to the subspace generated by $\{U_T : T \in \mathcal{K}, T \neq \emptyset\}$. That is, if for $T \in \mathcal{K}^*$ and $K \in \mathcal{P}(N)$ we let

  $$\bar{U}_T(K) = \begin{cases} 1 & \text{if } T \subset K, \\ 0 & \text{otherwise,} \end{cases}$$

  and for $v = \sum_{T \in \mathcal{K}^*} a_T \, U_T$ we let $\bar{v} = \sum_{T \in \mathcal{K}^*} a_T \, \bar{U}_T$, which is defined on $\mathcal{P}(N)$, then $\phi^U(v) = \phi^S(\bar{v})$.

- Hence, for any $v \in \mathcal{V}$, we may find a (unique) potential for $\phi^U(v)$ along the lines of Hart and Mas-Colell (1989), looking at the potential for $\bar{v}$.
- Hart and Mas-Colell (1989, p. 605) consider also a monotonicity property, which ϕ^U satisfies.
- In general, ϕ^U is not marginalist in the sense of (1.4), since it does not satisfy the null player axiom (consider $\mathcal{K} = \{\emptyset, \{1\}, N = \{1,2\}\}$ and $v = U_N$). However, through the extensions mentioned above, we do obtain a different kind of marginalism.

Let us give now some examples in which ϕ^U is marginalist.

Example 3.2. Consider $\mathcal{K} = \{\emptyset, K_1 = \{1\}, K_2 = \{2,3\}, N = \{1,2,3\}\}$, whose maximal chains are $W_1 = (\emptyset, K_1, N)$ and $W_2 = (\emptyset, K_2, N)$.

ϕ^U is marginalist, since we may write it as

$$\phi_i^U(v) = \frac{2}{3}\,\psi_{W_1}(v) + \frac{1}{3}\,\psi_{W_2}(v).$$

However, it does not fit in any of the models 1, 2, or 3 of Section 2.1, since both chains have length 2, but W_1 has weight $2/3$ and W_2 has weight $1/3$. It does not fit the entropy model 4 either, since both chains have an arc of length 1 and another of length 2, and hence equal entropy. Lastly, it does not fit the potential model of Section 2.2, since summing the flows along W_1 we obtain $4/3$, and summing the flows along W_2 we obtain $2/3$.

Since players 2 and 3 are equivalent in $\mathcal{K}$, we may consider $\overline{N} = \{\bar{1}, \bar{2}\}$, identify $\bar{1}$ with $\{1\}$, $\bar{2}$ with $\{2,3\}$, $\mathcal{P}(\overline{N})$ with $\mathcal{K}$, and any game $\bar{v}$ on $\mathcal{P}(\overline{N})$ with a game v on $\mathcal{K}$. Hence, if Φ is the Shapley value on $\overline{N}$, it is natural to define ϕ on $\mathcal{K}$ by

$$\phi_1(v) = \Phi_{\bar{1}}(\bar{v}), \quad \phi_2(v) = \phi_3(v) = \frac{1}{2}\,\Phi_{\bar{2}}(\bar{v}).$$

Thus, $\phi_2^U(U_N) = \phi_3^U(U_N) = 1/3$, whereas $\phi_2(U_N) = \phi_3(U_N) = 1/4$.

The previous example is easily generalized. If $\sim$ denotes the equivalence relation between players, we may consider any set N and $\mathcal{K}$ so that if $\overline{N} = N/{\sim}$, then $\mathcal{K}$ is isomorphic to $\mathcal{P}(\overline{N})$. In this case, ϕ^U is marginalist, and ϕ^S on $\overline{N}$ induces a value on $\mathcal{K}$ which differs from ϕ^U.

It is not the case that whenever ϕ^U is marginalist then $\mathcal{K}$ is isomorphic to $\mathcal{P}(\overline{N})$ for some $\overline{N}$, as may be readily seen by taking $n = 3$, $\mathcal{K} = \{\emptyset, \{1,2\}, \{1,3\}, \{2,3\}, N\}$, in which there are no equivalent players and $|\mathcal{K}| = 5$. ♢

We will find it useful to refer to the following equation: if ϕ^U is marginalist and is written in the form (1.4), for $T \in \mathcal{K}^*$ and $i \in T$ we have

$$\phi_i^U(U_T) = \sum_{\substack{(K,K') \in \mathcal{A}_i \\ T \not\subset K,\, T \subset K'}} \lambda_i(K,K'), \tag{3.1}$$

with the understanding that a sum is zero if it has no terms.

Lemma 3.3. *Suppose ϕ^U may be written in the form (1.4), and either of the following is satisfied:*

H-1. $\lambda_i(K, K') \geq 0$ *for all* $(K, K') \in \mathcal{A}_i$,

H-2. for each $T \in \mathcal{K}^*$ *and* $i \in T$, *there exists at most one arc of the form* (T', T) *in* $\mathcal{A}_i$.

Then,

(i) for all $T \in \mathcal{K}^*$ *there exists a constant* $\alpha(T) > 0$ *such that*

$$\sum_{K: T \subset K} \alpha(K) = \frac{1}{|T|},$$

(ii) $\lambda_i(K, T) = \alpha(T)$ *for all* $i \in T$ *and* $(K, T) \in \mathcal{A}_i$,

(iii) for all $K \in \mathcal{K}^*$ *and* $i \in K$, *there exists a unique arc* $(K', K) \in \mathcal{A}_i$.

Remark 3.4. H-1 is a condition on the coefficients, whereas H-2 is on $\mathcal{K}$.

Notice also that if $\mathcal{K} = \mathcal{P}(N)$, (1.3) yields

$$\alpha(T) = \frac{(|T| - 1)!\,(n - |T|)!}{n!} \qquad \text{for } T \neq \emptyset. \qquad \diamondsuit$$

Proof of Lemma 3.3. We first observe that under H-1 or H-2, (i) and (ii) imply (iii).

Now, $\phi_i(U_N) = 1/n$ for all $i \in N$, and (3.1) implies that $N \in \mathcal{K}_i^-$. Taking $\alpha(N) = 1/n$, the result follows if $(\emptyset, N) \in \mathcal{A}$. Otherwise, for fixed $i \in N$, let $(R, N) \in \mathcal{A}_i$ (and hence, $i \notin R$). Applying (3.1) for $T = N$ and $T = R$ we obtain

$$\lambda_i(R, N) + \sum_{\substack{(R', N) \in \mathcal{A}_i \\ R' \neq R}} \lambda_i(R', N) = \frac{1}{n} \quad \text{and} \quad \sum_{\substack{(R', N) \in \mathcal{A}_i \\ R' \neq R}} \lambda_i(R', N) = 0, \tag{3.2}$$

and therefore $\lambda_i(R, N) = 1/n$. Since this holds for any $(R, N) \in \mathcal{A}_i$, then there is only one such arc (otherwise the first sum in (3.2) would exceed $1/n$). So the result holds for $T = N$.

Let $T \in \delta^-(N)$. For $i \in T$ we have

$$\begin{aligned} \frac{1}{|T|} = \phi_i^U(U_T) &= \sum_{\substack{(K, K') \in \mathcal{A}_i \\ T \not\subset K, T \subset K'}} \lambda_i(K, K') \\ &= \sum_{(K, T) \in \mathcal{A}_i} \lambda_i(K, T) + \sum_{\substack{(K, N) \in \mathcal{A}_i \\ T \not\subset K}} \lambda_i(K, N) = \sum_{(K, T) \in \mathcal{A}_i} \lambda_i(K, T) + \frac{1}{n}, \end{aligned}$$

and we define

$$\alpha(T) = \frac{1}{|T|} - \frac{1}{n}.$$

Thus, $\delta^-(N) \subset A$, where A is subset of coalitions $T \in \mathcal{K}$ such that:

- for all $K' \in \mathcal{K}^*$ with $T \subset K'$, there holds $\sum_{R:K'\subset R} \alpha(R) = 1/|K'|$, and

$$\alpha(K') = \sum_{(K,K')\in\mathcal{A}_i} \lambda_i(K,K') > 0 \quad \text{for all } i \in K',$$

- for all K' with $T \subsetneq K'$ and all $i \in K'$, there holds $K' \in \mathcal{K}_i^-$, and

$$\alpha(K') = \lambda_i(K,K') > 0 \quad \text{if } (K,K') \in \mathcal{A}_i \text{ and } T \subset K.$$

Let $S \in \mathcal{K}$ be such that $\{T \in \mathcal{K} : S \subsetneq T\} \subset A$, and let us show that necessarily $S \in A$. To do so, we consider the following alternatives for S:

(a) If $S = \emptyset$, we only have to verify the second part of the definition of A. If $T \in \delta^+(\emptyset)$, there is only one arc ending in T, and we must have $\alpha(T) = \lambda_i(\emptyset, T) > 0$ for all $T \in \delta^+(\emptyset)$ and all $i \in T$. Thus, $S = \emptyset \in A$.

(b) If $S \neq \emptyset$, (3.1) yields

$$\phi_i^U(U_S) = \sum_{(K,S)\in\mathcal{A}_i} \lambda_i(K,S) + \sum_{K':S\subsetneq K'} \sum_{\substack{(K,K')\in\mathcal{A}_i \\ S\not\subset K}} \lambda_i(K,K').$$

If $i \in S$, for K' such that $S \subsetneq K'$ there are no arcs $(K,K') \in \mathcal{A}_i$ with $S \subset K$. Since, furthermore, $K' \in A$, then $K' \in \mathcal{K}_i^-$, and

$$\sum_{\substack{(K,K')\in\mathcal{A}_i \\ S\not\subset K}} \lambda_i(K,K') = \sum_{(K,K')\in\mathcal{A}_i} \lambda_i(K,K') = \alpha(K').$$

Hence,

$$\phi_i^U(U_S) = \frac{1}{|S|} = \sum_{(K,S)\in\mathcal{A}_i} \lambda_i(K,S) + \sum_{T:S\subsetneq T} \alpha(T),$$

and

$$\alpha(S) = \frac{1}{|S|} - \sum_{T:S\subsetneq T} \alpha(T) = \sum_{(K,S)\in\mathcal{A}_i} \lambda_i(K,S),$$

is independent of $i \in S$.

If $i \notin S$,

$$\begin{aligned} 0 = \phi_i^U(U_S) &= \sum_{K':S\subsetneq K'} \sum_{\substack{(K,K')\in\mathcal{A}_i \\ S\not\subset K}} \lambda_i(K,K') \\ &= \sum_{(S,K')\in\mathcal{A}_i} (\alpha(K') - \lambda_i(S,K')) + \sum_{\substack{K':S\subsetneq K' \\ K'\notin\delta^+(S)}} \sum_{\substack{(K,K')\in\mathcal{A}_i \\ S\not\subset K}} \lambda_i(K,K'). \end{aligned}$$

Then, if either H-1 or H-2 holds, the last sum on the right vanishes, and $\alpha(K') = \lambda_i(S,K')$ for all $(S,K') \in \mathcal{A}_i$.

The equality of incoming and outgoing flows through S imply

$$\sum_{i:S\in\mathcal{K}_i^+}\alpha(S)=\alpha(S)\,|\{i:S\in\mathcal{K}_i^-\}|$$

$$=\sum_{(S,K)\in\mathcal{A}}\sum_{i:(S,K)\in\mathcal{A}_i}\lambda_i(S,K)=\sum_{(S,K)\in\mathcal{A}}\alpha(K)\,|i:(S,K)\in\mathcal{A}_i|$$

and so $\alpha(S)>0$. Hence, $S\in A$.

Suppose now there exists $S\in\mathcal{K}$, $S\neq N$ such that $S\notin A$. Then there exists $T\in\delta^+(S)$ such that $T\notin A$. Substituting T for S, and continuing in this way, we reach a contradiction, as $\delta^-(N)\subset A$.

Thus, $A=\{T\in\mathcal{K}:T\neq N\}$, and the result follows. □

Theorem 3.5. *If ϕ^U is marginalist, and either H-1 or H-2 is satisfied, then $\mathcal{K}$ satisfies the following property:*

P-1. If $R,T\in\mathcal{K}^*$ and $T\subset R$, then $\{R\setminus S:(S,R)\in\mathcal{A},T\not\subset S\}$ is a partition of T.

Proof. Using Lemma 3.3, we see that if $T\in\mathcal{K}^*$, $(S,R)\in\mathcal{A}$, $T\not\subset S$, and $T\subset R$, then $R\setminus S\subset T$, since if $j\in R\setminus S$ and $j\notin T$, we would have

$$\phi_j^U(U_T)=0=\sum_{\substack{(K,K')\in\mathcal{A}_j\\ T\not\subset K,T\subset K'}}\lambda_j(K,K')=\sum_{\substack{(K,K')\in\mathcal{A}_j\\ T\subsetneq K'}}\alpha(K')\geq\alpha(R)>0.$$

Therefore, if $T\subset R$, $\bigcup_{\substack{S:S\neq T\\ (S,R)\in\mathcal{A}}}R\setminus S\subset T$, and the union is disjoint, since if $i\in R$, there is only one arc of the form (S,R) in $\mathcal{A}_i$. Finally, by Lemma 3.3, for all $i\in T\subset R$ there exists an arc $(S,R)\in\mathcal{A}_i$. □

As regular families satisfy H-2, we have:

Corollary 3.6. *If $\mathcal{K}$ is regular, then ϕ^U is marginalist if and only if $\mathcal{K}=\mathcal{P}(N)$.*

The following is a converse to Theorem 3.5:

Theorem 3.7. *If $\mathcal{K}$ satisfies P-1, then ϕ^U is marginalist.*

Proof. Let us define $\alpha:\mathcal{K}^*\to\mathbb{R}$ recursively by setting $\alpha(N)=1/n$, and for $S\in\mathcal{K}^{**}$,

$$\sum_{K:(S,K)\in\mathcal{A}}|K\setminus S|\,\alpha(K)=|S|\,\alpha(S),\tag{3.3}$$

so that $\alpha(S)>0$ for all $S\in\mathcal{K}^*$. We will show that $\Lambda:\mathcal{A}\to\mathbb{R}$ defined by

$$\Lambda(K,K')=|K'\setminus K|\,\alpha(K'),$$

is a flow of value 1.

The conservation of flow through $S \in \mathcal{K}^{**}$ is represented by the equality

$$\sum_{K:(S,K)\in\mathcal{A}} \Lambda(S,K) = \sum_{K:(K,S)\in\mathcal{A}} \Lambda(K,S),$$

which written in terms of α is equivalent to (3.3), given that P-1 is satisfied:

$$\sum_{K:(S,K)\in\mathcal{A}} |K \setminus S|\, \alpha(K) = \sum_{K:(K,S)\in\mathcal{A}} |S \setminus K|\, \alpha(S) = |S|\, \alpha(S),$$

Also, since $\sum_{(K,N)\in\mathcal{A}} \Lambda(K,N) = |N|/n = 1$, the value of the flow Λ is 1.

By Theorem 2.9 we obtain a marginalist value ϕ associated with Λ, with

$$\lambda_i(K,K') = \frac{\Lambda(K,K')}{|K' \setminus K|} = \alpha(K') > 0 \quad \text{for all } (K,K') \in \mathcal{A}_i.$$

For $T \in \mathcal{K}^*$ and $i \in N$ we have

$$\phi_i(U_T) = \sum_{\substack{(K,K')\in\mathcal{A}_i\\ T\not\subset K, T\subset K'}} \alpha(K').$$

If $i \notin T$, the sum is 0 since the arcs (K,K') with $T \not\subset K$ and $T \subset K'$ form a partition of T, and the sum has no terms. If $i \in T$, $\phi_i(U_T) = 1/|T|$, as the flow through the cut $\{(K,K') : T \not\subset K, T \subset K'\}$ is

$$\begin{aligned} 1 &= \sum_{\substack{(K,K')\in\mathcal{A}\\ T\not\subset K, T\subset K'}} \Lambda(K,K') = \sum_{K':T\subset K'} \alpha(K') \sum_{\substack{(K,K')\in\mathcal{A}\\ T\not\subset K}} |K' \setminus K| \\ &= \sum_{K':T\subset K'} \alpha(K')\, |T| = |T|\, \phi_i(U_T). \end{aligned}$$

Thus, $\phi^U = \phi$. □

We cannot relax the condition P-1 to a simpler one such as

P-2. for all $T \in \mathcal{K}^*$, $\{T \setminus S : (S,T) \in \mathcal{A}\}$ is a partition of T,

and still have ϕ^U marginalist as the following example shows:

Example 3.8. Let $n = 3$ and $\mathcal{K} = \{\emptyset, K_1 = \{1\}, K_2 = \{2\}, K_3 = \{3\}, K_4 = \{1,2\}, N\}$, which satisfies P-2.

If ϕ^U were marginalist, the values of the flow Λ arriving to N would be determined, so that $\Lambda(\emptyset, K_3) = \Lambda(K_3, N) = 2/3$, and $\Lambda(K_4, N) = 1/3$. Also, by symmetry, $\Lambda(\emptyset, K_1) = \Lambda(K_1, K_4) = \Lambda(\emptyset, K_2) = \Lambda(K_2, K_4) = 1/6$. If $T = K_1$, then $\phi_2(U_T) = 1/3$ (even though $\phi_1(U_T) = 1$ and $\phi_3(U_T) = 0$). ♢

4 Concluding remarks

There are several ways of defining a value similar to Shapley's for general subfamilies of coalitions, and we have presented two frameworks for doing so. Although if $\mathcal{K} = \mathcal{P}(N)$ the set of axioms we used are equivalent, in general we obtain values with quite different properties.

If we choose **N**, **L** and **E**, we obtain a marginalist value, and a rich mathematical theory behind, but we still need an extra property (or axiom) to obtain uniqueness, for which we have put forward several possibilities. In contrast, **L** and **EU** give a unique value, ϕ^U, which is efficient, but need not be marginalist (which is not surprising, since **L** and **EU** together do not imply **N**). Interestingly, the dissimilarities are paralleled by the choice of linear bases of $\mathcal{V}$: whereas in the former case we use only elementary games for our proofs, in the latter—by the very definition—we use unanimity games.

ϕ^U has received little attention for general subfamilies, perhaps because it is not necessarily marginalist. Even when marginalist, ϕ^U challenges the marginalist approach. More precisely, what is the "fair" value for the players in Example 3.2 when $v = U_N$?: ϕ^U assigns $1/3$ to each, whereas we may think that since 2 and 3 act as a unit, player 1 should receive $1/2$, and players 2 and 3 should receive $1/4$ each.

We have not pursued the fairness subject any further, as this paper is mathematically oriented, and its only purpose is to show some possibilities.

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