Aproximaciones numéricas en dominios no estándar.

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1) G. Acosta, M. G. Armentano, R. G. Durán and A. L. Lombardi, *"Nonhomogeneous Neumann problem for the Poisson equation in domains with an external cusp"*, Journal of Mathematical Analysis and Applications 310(2), 397-411, 2005.

2) G. Acosta, M. G. Armentano, R. G. Durán and A. L. Lombardi, *Finite Element Approximations in a non-Lipchitz domain*, SIAM J. Num. Anal. 45, 277-295, 2007.

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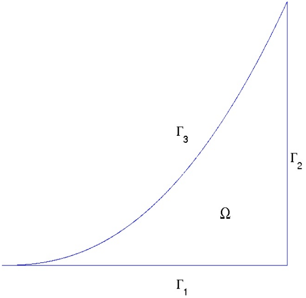
Consideremos

$$\Omega = \{(x,y) : 0 < x < 1,\ 0 < y < x^{\alpha},\ \alpha > 1\},$$



con frontera $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

Hallar u tal que:

$$\begin{cases} -\Delta u & = f, \quad \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} & = g, \quad \text{on} \quad \Gamma_3 \\ \frac{\partial u}{\partial \nu} & = 0, \quad \text{on} \quad \Gamma_1 \\ u & = 0, \quad \text{on} \quad \Gamma_2 \end{cases}$$

con ν la normal exterior.

Notación:

$$\phi(x) := x^{\alpha}$$

$$\Gamma_3 := \{(x, \phi(x)), x \in (0,1)\}$$

Sea $V = \{v \in H^1(\Omega) : v|_{\Gamma_2} = 0\}$.

El problema variacional es: Hallar $u \in V$ tal que

$$a(u, v) = L_1(v) + L_2(v) \qquad \forall v \in V,$$

donde

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \quad , \quad L_1(v) = \int_\Omega fv \quad \text{and}$$

$$L_2(v) = \int_{\Gamma_3} gv.$$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v$$

$a(u,v)$ es continua y coerciva (Poincaré) en V y si por ejemplo $f \in L^2(\Omega)$,

$$L_1(v) = \int_\Omega fv$$

es continua en $H^1(\Omega)$.

El problema entonces es manejar el término

$$L_2(v) = \int_{\Gamma_3} gv.$$

usualmente para $g \in L^2$, Schwartz + teorema de traza.

Problema: Ω no es Lipschitz no vale el teorema estándar de trazas para $H^1(\Omega)$.

Considerar p.ej. $\phi(x) = x^\alpha$, $\alpha > 1$, y la función $u(x,y) = x^{-\gamma}$:

$$u \in H^1(\Omega) \qquad \text{iff} \qquad \gamma < \frac{\alpha - 1}{2}$$

sin embargo

$$u \in L^2(\Gamma) \qquad \text{iff} \qquad \gamma < \frac{1}{2}$$

en particular si $\alpha > 2$ obtenemos funciones de $H^1(\Omega)$ que no estan en $L^2(\partial\Omega)$.

...entonces para $g \in L^2(\Gamma_3)$, la continuidad en $H^1(\Omega)$ de L_2

$$L_2(v) = \int_{\Gamma_3} gv.$$

no puede manejarse de ese modo.

Mazya, Netrusov, Poborchi (2000) resultados de trazas para este tipo de dominios.

En particular

$$\|vx^{\frac{\alpha}{2}}\|_{L^2(\Gamma)} \leq C\|v\|_{H^1(\Omega)}$$

Entonces se puede escribir

$$|L_2(v)| \leq \int_{\Gamma_3} |gx^{-\frac{\alpha}{2}} vx^{\frac{\alpha}{2}}| \leq \|gx^{-\frac{\alpha}{2}}\|_{L^2(\Gamma_3)}\|vx^{\frac{\alpha}{2}}\|_{L^2(\Gamma_3)} \leq C\|v\|_{H^1}.$$

usando Lax-Milgram :

Theorem Si $gx^{-\frac{\alpha}{2}} \in L^2(\Gamma_3)$ y $f \in L^2(\Omega)$ existe una única solución $u \in V$ de nuestro problema.
Sin embargo

$$gx^{-\frac{\alpha}{2}} \in L^2(\Gamma_3)$$

implica que g se anula en el origen.

Otra variante es

$$|L_2(v)| \leq \int_{\Gamma_3} |gv| \leq \|g\|_{L^q(\Gamma_3)} \|v\|_{L^p(\Gamma_3)} \leq C\|v\|_{H^1}.$$

y el problema entá en hallar condiciones suficientes en u para tener trazas en L^p

Mostramos:

$$\|u\|_{L^p(\Gamma)} \leq C\left(\|u\phi^{-\frac{1}{p}}\|_{L^p(\Omega)} + \|\nabla u\phi^{\frac{p-1}{p}}\|_{L^p(\Omega)}\right)$$

(tambien se puede repartir peso y obtener el resultado de Mazya et al.)
Técnica:

$$\psi = \frac{1}{\phi'(x)} \qquad \eta = \frac{y}{\phi(x)}$$

Esto + inmersión en cúspides (Adams, 1975)

$$H^1(\Omega) \subset L^r \qquad r \leq \frac{2(\alpha+1)}{\alpha-1}$$

da

Teorema Sea $u \in H^1(\Omega)$, y $1 \leq p \leq 2$. Si $\alpha < 1 + \frac{2}{p}$ entonces $u \in L^p(\Gamma)$ y

$$\|u\|_{L^p(\Gamma)} \leq C\|u\|_{H^1(\Omega)}.$$

El resultado es sharp en el sentido de que para $\alpha > 2$ hay funciones en H^1 tales que la traza no está en L^2.

Recopilando:

Teorema Sea $1 \leq p \leq 2$, $g \in L^q(\Gamma_3)$ con $q = \frac{p}{p-1}$, and $f \in L^2(\Omega)$. Si $\alpha < 1 + \frac{2}{p}$ entonces existe solución única $u \in V$ de nuestro problema.

Más regularidad Sobolev es posible?.

Probamos, bajo ciertas condiciones en los datos:

La solución está en $H^2(\Omega)$

Theorem Sea u la solución de nuestro problema, $f \in L^2(\Omega)$, y g tal que, si $h(t) := g(t, t^\alpha)$, $ht^{-\frac{\alpha}{2}} \in L^2(0,1)$ y $h' t^{1-\frac{\alpha}{2}} \in L^2(0,1)$. Entonces, $u \in H^2(\Omega)$ y existe una constante C dependiendo solo de α tal que

$$\|u\|_{H^2(\Omega)} \leq C \left\{ \|f\|_{L^2(\Omega)} + \|ht^{-\frac{\alpha}{2}}\|_{L^2(0,1)} + \|h' t^{1-\frac{\alpha}{2}}\|_{L^2(0,1)} \right\}$$

Observación

La condición

$$ht^{-\frac{\alpha}{2}} \in L^2(0,1)$$

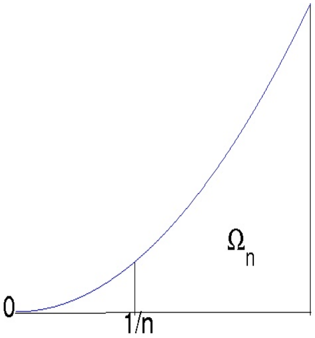
es casi óptima, de hecho mostramos que si $u \in H^2$,

$$\int \frac{h^r}{t^{\alpha(r-1)}} < \infty$$

$\forall\ r < 2$.

La otra condicion??? ... no sabemos.

Idea de la demostración: Khelif (1978) y Grisvard (1985) caso homogeneo
Trabajar con dominios "truncados"



obtener la estimación:

$$\|u\|_{H^2(\Omega_n)} \leq C_n \left\{ \|f\|_{L^2(\Omega_n)} + \|h t^{-\frac{\alpha}{2}}\|_{L^2(1/n,1)} + \|h' t^{1-\frac{\alpha}{2}}\|_{L^2(1/n,1)} \right\}$$

y ver que C_n se acota independientemente de n

El control en H^1 sale directo.
Para las derivadas segundas:

Si θ y ψ estan en $H^1(\Omega_n)$

$$\int_{\Omega_n} \theta_x \psi_y = \int_{\Omega_n} \theta_y \psi_x + \int_{\partial\Omega_n} \psi \frac{\partial \theta}{\partial \tau}$$

$$\theta = \frac{\partial u_n}{\partial x} \qquad \text{and} \qquad \psi = \frac{\partial u_n}{\partial y}$$

$$\begin{aligned}
\int_{\Omega_n} f^2 = \int_{\Omega_n} (\Delta u_n)^2 &= \int_{\Omega_n} (\theta_x + \psi_y)^2 \\
&= \int_{\Omega_n} \theta_x^2 + 2\int_{\Omega_n} \theta_x \psi_y + \int_{\Omega_n} \psi_y^2 \\
&= \int_{\Omega_n} \theta_x^2 + 2\int_{\Omega_n} \theta_y \psi_x + \int_{\Omega_n} \psi_y^2 + 2\int_{\partial\Omega_n} \psi \frac{\partial \theta}{\partial \tau} \\
&= |u_n|^2_{H^2(\Omega_n)} + 2\int_{\partial\Omega_n} \psi \frac{\partial \theta}{\partial \tau},
\end{aligned} \tag{1}$$

donde $|u_n|_{H^2(\Omega_n)}$ es la seminorma de u_n in $H^2(\Omega_n)$.

Como tenemos regularidad H^2 esperamos orden óptimo en aproximaciones lineales por elementos finitos, i.e.:

1. Orden 2 para el error en norma L^2.
2. Order 1 para el error en la norma de la energía.

Remark

Nos restringimos a $\alpha < 3$: Adams (1975)

$$H^2(\Omega) \subset L^\infty(\Omega) \qquad \text{if} \qquad \alpha < 3.$$

Aproximacion por elementos finitos.

THE RECIPE:

a) Find an explicit H^2 solution of our problem with a fixed cusp, for instance with $\alpha = 2$.
b) Approximate the solution by standard finite linear elements.
c) Check the error.

a) THE H^2 SOLUTION

Consider

$$u(x,y) = x^s - 1$$

which is in $H^2(\Omega)$ if $s > \frac{1}{2}$. For this choice

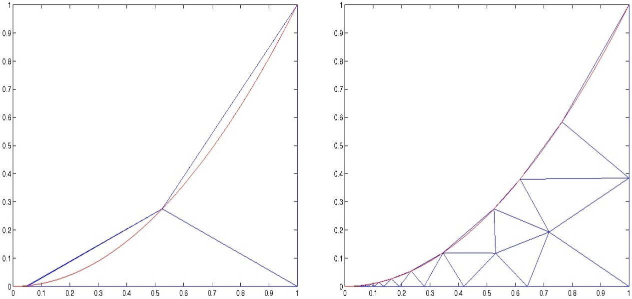
$$f(x,y) = s(s-1)x^{s-2}$$

and

$$h(t) = g(t,t^2) = \frac{-2st^s}{\sqrt{1+4t^2}}.$$

b) THE F.E.M. APPROXIMATION

Triangulate the domain Ω



(Observe that the triangulation $\mathcal{T}_h$ defines a polygonal domain Ω_h and $\Omega \subset \Omega_h$).

Apply the standard finite linear element method:

Find $u_h \in V_h$

$$V_h = \{v_h \in V : \; v_h|_T \in \mathcal{P}_1 \quad \forall T \in \mathcal{T}_h : v_h|_{\Gamma_2} = 0\}$$

such that

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v_h = \int_{\Omega} f v_h + \int_{\partial\Omega_{h,3}} I_h(g) v_h \qquad \forall v_h \in V_h.$$

(I_h denotes the piecewise linear interpolation at the vertices of the triangulation $\mathcal{T}_h$ which lies on Γ_3).

c) COMPUTE THE ERROR

value of s ($\alpha = 2$)	L^2 order in h	L^2 order in number of nodes
0.55	1.65818970125595	0.86088757665489
0.60	1.70694615889165	0.88620062052947
0.65	1.75601497954524	0.91167583489716
0.70	1.80518355660826	0.93720284011453
0.75	1.85424176187600	0.96267254325886
0.80	1.90295927796932	0.98796536973008
0.85	1.95106069486770	1.01293833403926
0.90	1.99819397485982	1.03740866765759
0.95	2.04388765319860	1.06113159874547

Recall that $u = x^s - 1 \in H^2$ if $s > \frac{1}{2}$.

So, the question is...

What is WRONG?

Observe the STRICT inclusion:

$$\Omega \subset \Omega_h$$

In general if $\Omega \neq \Omega_h$ some "non - conforming" like term appears in the error equation.

For SMOOTH Ω the non-conforming terms can be properly bounded (Bramble and King (1994)) and, as a consequence, optimal order convergence is obtained.

The proof given by Bramble and King relies strongly on EXTENSION THEOREMS in H^2.

Our Ω is NOT an extension domain (Peter Jones (1981)).

Considering again $u(x,y) = x^s - 1 \in H^2(\Omega)$.

If $s < 1$ the function IS NOT in $H^2(\Omega_h)$, for any h.

value of s ($\alpha = 2$)	L^2 order in h	L^2 order in number of nodes
0.55	1.65818970125595	0.86088757665489
0.60	1.70694615889165	0.88620062052947
0.65	1.75601497954524	0.91167583489716
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0.90	1.99819397485982	1.03740866765759
0.95	2.04388765319860	1.06113159874547

We need a new type of extension theorem.

Let $H^2_\alpha(\mathcal{D})$ be the weighted Sobolev space defined by:

$$H^2_\alpha(\mathcal{D}) = \{v : r^{\frac{\alpha-1}{2}} D^\gamma v \in L^2(\mathcal{D}), \text{ for any } \gamma, |\gamma| \leq 2\}$$

$(r = \sqrt{x^2 + y^2})$ with the norm

$$\|u\|_{H^2_\alpha(\mathcal{D})} = \sum_{|\gamma|\leq 2} \|r^{\frac{\alpha-1}{2}} D^\gamma v\|_{L^2(\mathcal{D})}$$

We showed (2007) that H^2 solutions of our problem can be extended to $H^2_\alpha(\mathbb{R}^2)$.

Also Maz'ya et. al, n years ago ... in russian!,

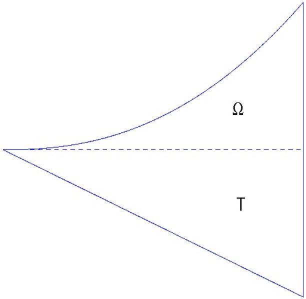
See the book Differentiable Functions on Bad Domains, 1997.

The extension theorem

Theorem Let u be the solution of our problem then there exists a function $\hat{u} \in H^2_\alpha(\mathbb{R}^2)$ such that $\hat{u} = u \quad$ in Ω and

$$\|\hat{u}\|_{H^2_\alpha(\mathbb{R}^2)} \leq C\|u\|_{H^2(\Omega)}$$

Proof First: Extend the solution to the triangle
$T = \{(x,y) : 0 \leq x \leq 1, -x \leq y \leq 0\}$



Second: Observe that $\mathcal{D} = \Omega \cup T$ is a Lipchitz domain. Show that $\hat{u}(x, y)$ belongs to the weighted Sobolev space $H^2_\alpha(\mathcal{D})$.

Third: Apply the results obtained by Seng-Kee Chua (1992) and extend the function to $\mathbb{R}^2$, if $\alpha < 3$ (again this condition!, the weight is in the class A_2).

It can be proved:

- $\hat{u} \in H^2_\alpha(\mathbb{R}^2) \cap H^1(\Omega_h)$
- $\hat{u} \in W^{2,p}(\Omega_h)$ for $p < \frac{4}{\alpha+1}$

The moral is:

Regular solutions u defined in Ω behave like SINGULAR SOLUTIONS since they can not be extended with the same regularity to Ω_h!

Question : Numerical remedy?

Answer: Some type of adaptivity.

For poligonal domains (i.e. $\Omega = \Omega_h$) but with angles greater than Π (and hence with singular solutions) Grisvard (1985) showed that solutions of elliptic problems with this kind of singularities (i.e. solutions belonging to the space H^2_α) can be approximated with optimal order by introducing a graded mesh.

His main tool: Sharp Lagrange interpolation estimates at the singularity.

$$\text{For all } v \in H^2_\alpha(\mathbb{R}^2) \cap H^1(\Omega_h)$$

$$|\nabla(v - \Pi_h(v))|^2_{L^2(K)} \leq C h_K^{3-\alpha} \sum_{|\beta|=2} \int_K |D^\beta v|^2 r^{\alpha-1}$$

$$\forall K \in \mathcal{T}_h$$

where $\Pi_h(v)$ denotes the linear Lagrange interpolation at the vertices of the triangle $K \in \mathcal{T}_h$.

And an appropriate mesh

1. Regularity assumption (i.e. the **minimum** angle condition).
2. Let h_K be the diameter of the triangle which has a vertex at $(0,0)$. There exists a constant C such that $h_K \leq C\, h^{\frac{2}{3-\alpha}}$
3. $h_K \leq C\, h \inf_K r^{\frac{\alpha-1}{2}}$ for all $K \in \mathcal{T}_h$ with no corner at $(0,0)$

where h is a parameter which goes to 0.

The idea is simple: One can write, for K such that $(0,0) \in K$

$$|\nabla(v - \Pi_h(v))|^2_{L^2(K)} \leq Ch^2 \sum_{|\beta|=2} \int_K |D^\beta v|^2 r^{\alpha-1}$$

and for the other triangles (usual Lagrange interpolation estimate)

$$|\nabla(v - \Pi_h(v))|^2_{L^2(K)} \leq Ch_K^2 \sum_{|\beta|=2} \int_K |D^\beta v|^2$$

and since $h_K \leq h \inf_K r^{\frac{\alpha-1}{2}}$

$$|\nabla(v - \Pi_h(v))|^2_{L^2(K)} \leq Ch^2 \sum_{|\beta|=2} \int_K |D^\beta v|^2 r^{\alpha-1}$$

So, for the Lagrange interpolation we have

$$|\nabla(v - \Pi_h(v))|^2_{L^2(K)} \leq Ch^2 \|u\|^2_{2,\alpha}$$

and since the FEM solution behaves like the best approximant, the same estimate is expected!.

Can we do the same?

The easy part:

Grisvard assumes that the mesh satisfies the usual regularity condition i.e, $h_K/\rho_K \leq \sigma$ which will not be satisfied in our case. However the interpolation estimate holds under the **maximum angle condition** (Apel (1999)).

The OTHER part:

Consistency terms! (since for us $\Omega \neq \Omega_h$).

We change the regularity assumption by the Maximum Angle Condition.

1. The maximum angle condition.
2. Let h_K be the diameter of the triangle which has a vertice in $(0,0)$. There exists a constant C such that $h_K \leq C\, h^{\frac{2}{3-\alpha}}$
3. $h_K \leq C\, h \inf_K x^{\frac{\alpha-1}{2}}$ for all $K \in \mathcal{T}_h$ with no corner at $(0,0)$

where h is a parameter which goes to 0.

(NOTE THAT $x \sim r$ in our domain)

Next: we show that such a mesh can be constructed:

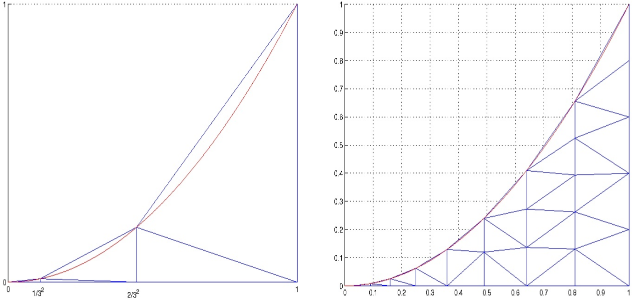
1) Divide the interval $[0,1]$ by using the points:

$$x_i = \left(\frac{i}{n}\right)^{\frac{2}{3-\alpha}} \qquad 1 \leq i \leq n$$

2) Take the points $(x_i, 0)$ in Γ_1, (x_i, x_i^{α}) in Γ_3 and divide the vertical lines in the uniform way.

Then the triangulation is obtained by using these points.

With such procedure the dimension of V_h is equivalent to n^2 with $h \sim 1/n$



And we prove:

Theorem Let u be the solution of our original Poisson problem (under the hypotheses of the regularity Theorem). Let τ_h be a mesh verifying the conditions required above, and u_h the linear finite element solution, then we have the following error estimate

$$\|u-u_h\|_{H^1(\Omega)} \leq Ch\sqrt{log(1/h)}\{\|f\|+\|ht^{-\frac{\alpha}{2}}\|_{L^2(0,1)}+\|h' t^{1-\frac{\alpha}{2}}\|_{L^2(0,1)}\}$$

For the same example considered above we have the following numerical results:

value of s ($\alpha = 2$)	L^2 order in h	L^2 order in number of nodes
0.55	2.00064455011318	1.11563231377340
0.60	2.00046659064628	1.11671041725597
0.65	2.00110950162586	1.11734684289910
0.70	2.00124353352113	1.11765054773069
0.75	2.00134533521139	1.11770400750771
0.80	2.00159423085932	1.11756955932164
0.85	2.00158981168176	1.11729440173258
0.90	2.00490538289029	1.11691442752246
0.95	2.00334871666207	1.11645710517780

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