El flujo de Ricci y sus aplicaciones

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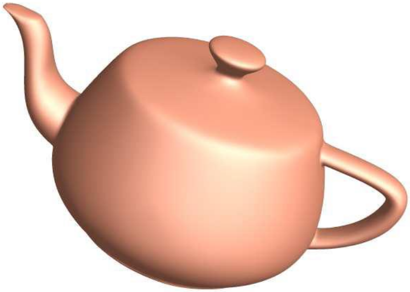


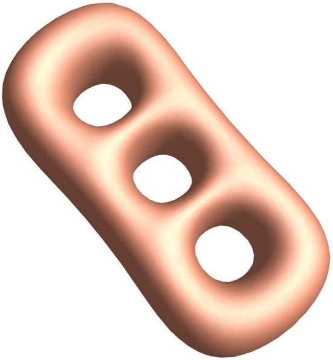
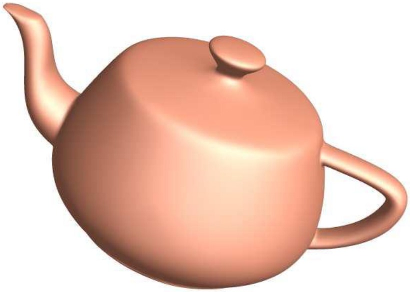


Figure: Calculus here? Why not?

Differentiable manifolds



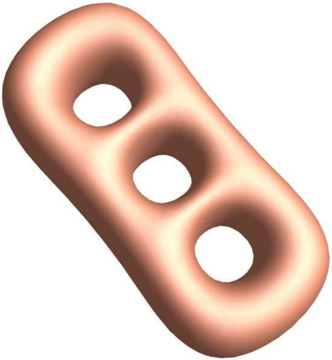


Figure: Calculus here? Why not? (pictures by David Gu, S-T Yau's talk)

Differentiable manifolds

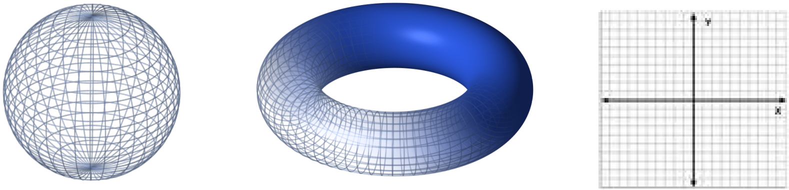
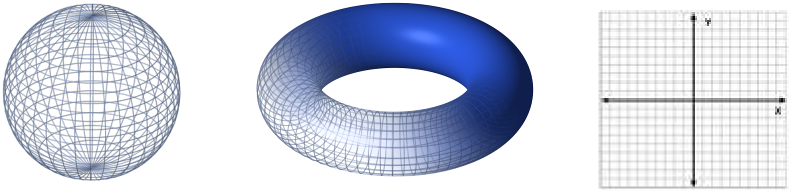


Figure: Looks locally as a Euclidean space $\mathbb{R}^n$, $n :=$ dimension (differentiable change of coordinates)

Differentiable manifolds



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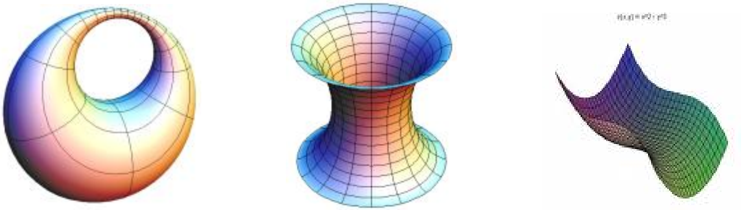

Figure: Other 2-dimensional manifolds or surfaces

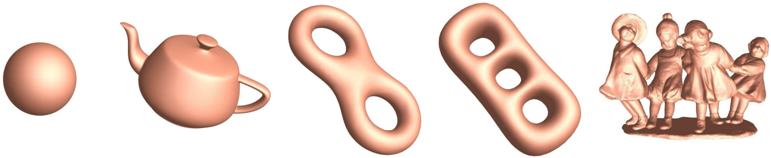

Figure: Clasificación de 2-variedades compactas (orientables), Siglo XIX

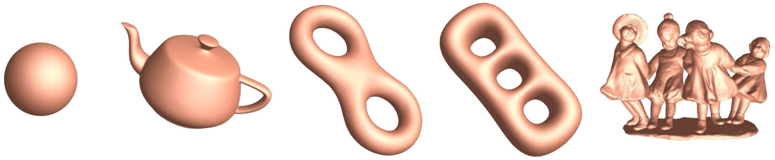

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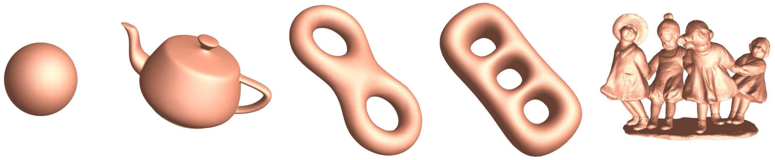
Figure: Variedad simplemente conexa



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Conjetura de Poincaré (1904): La 3-esfera es la única 3-variedad (orientable) compacta y simplemente conexa.

Ejemplos de 3-variedades compactas?

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Figure: Universo?

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Figure: Tierra: 2-variedad compacta?

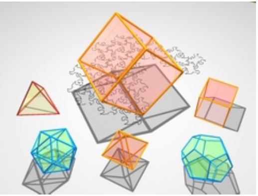
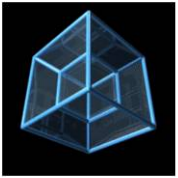
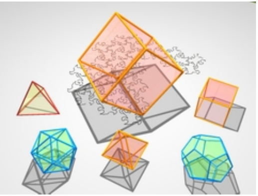
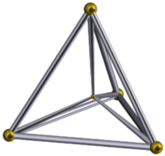
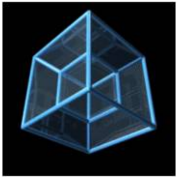
  

Figure: Proyecciones sobre $\mathbb{R}^3$ del simplex (o Pentachoron): 5 vértices, 10 aristas, 10 caras (triángulos) y 5 caras 3D (tetraedros); y del hipercubo: 16 vértices, 32 aristas, 24 caras (cuadrados) y 8 caras 3D (cubos).

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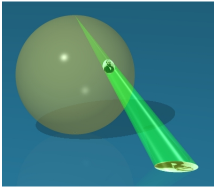
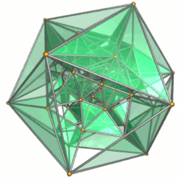



Figure: Proyección estereográfica sobre $\mathbb{R}^3$ del 24 (o Icositetrachoron): 24 vértices, 96 aristas, 96 caras (triángulos) y 24 caras 3D (octaedros).

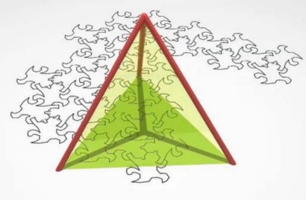
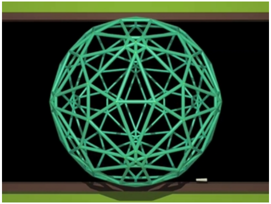
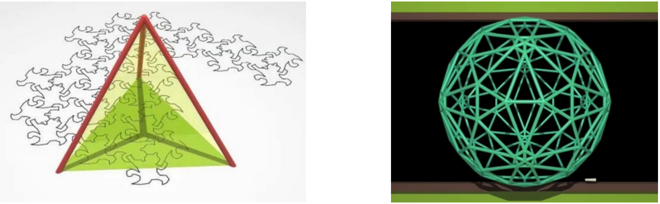
 

Figure: Intersección con $\mathbb{R}^3$ del 600 (o hexacosichoron), poliedro regular en dimensión 4 que tiene: 120 vértices, 720 aristas, 1200 caras triangulares y 600caras 3D (tetraedros).



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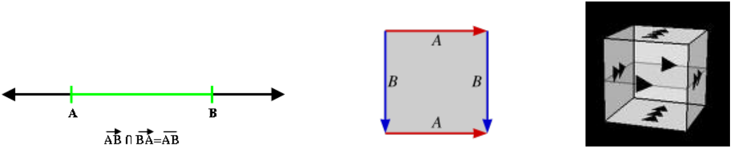


Figure: Identificación. 3-toro: T^3.

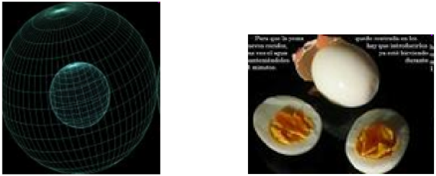
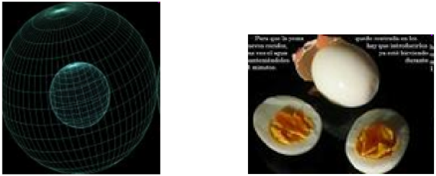


Figure: $S^2 \times S^1$



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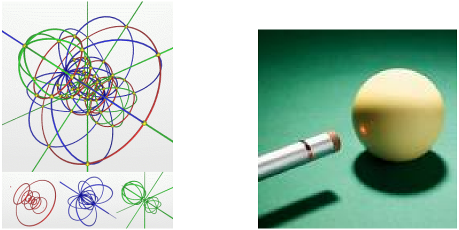


Figure: Proyección estereográfica de los paralelos, meridianos e hipermeridianos de la 3-esfera S^3: identificar todo el borde de una bola a un punto

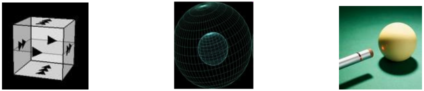
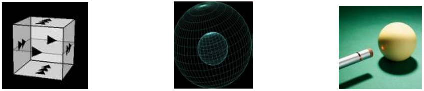


Figure: Ejemplos de 3-variedades: T^3, $S^2 \times S^1$, S^3.



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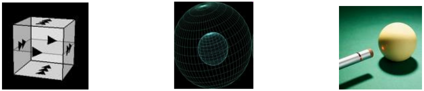
Conjetura de Poincaré (1904): La 3-esfera es la única 3-variedad (orientable) compacta y simplemente conexa



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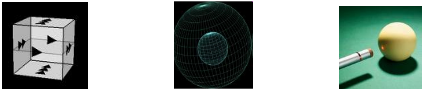

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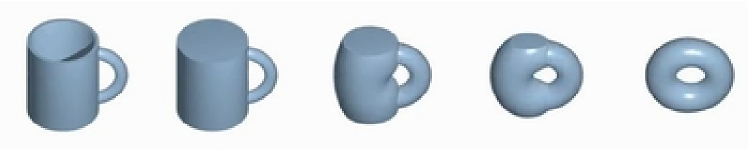

Figure: Deformación. Evolución geométrica.

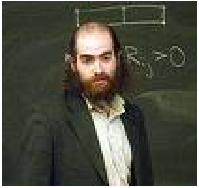

Figure: Henri Poincaré, Richard Hamilton, Grisha Perelman

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Riemannian manifolds

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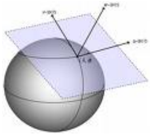

Figure: Tangent space at a point of a manifold (vector space)

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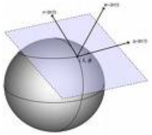
Riemannian metric on a manifold M:

Riemannian manifolds



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Riemannian metric on a manifold M: inner product g_x on each $\mathrm{T}_x M$

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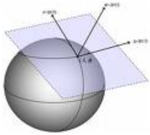

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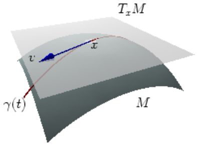
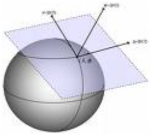


Figure: $g \rightsquigarrow$ length of tangent vectors and angles between them

Riemannian manifolds



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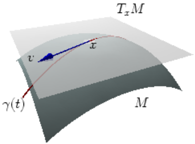

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Examples. Submanifolds of $\mathbb{R}^n$:

Riemannian manifolds

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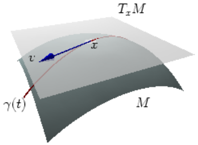

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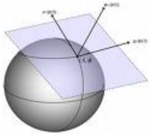
Riemannian manifolds



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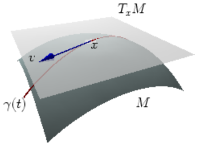

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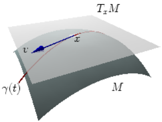
Examples. Submanifolds of $\mathbb{R}^n$: $\mathrm{T}_p M \subset \mathrm{T}_p \mathbb{R}^n \equiv \mathbb{R}^n$, $g_p \equiv \langle\cdot,\cdot\rangle|_{\mathrm{T}_p M}$.

(M, g): Riemannian manifold



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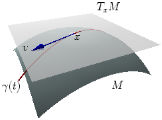

- length of curves:

$$\alpha : [a, b] \longrightarrow M,$$

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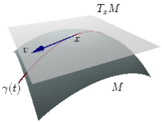

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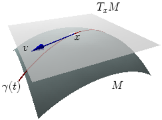

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$$d_g(p, q) := \inf\{\text{length}(\alpha) : \alpha \text{ curve from } p \text{ to } q\}$$

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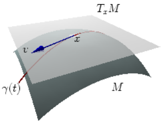

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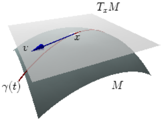

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- angle between curves at a crossing point
- area and volume

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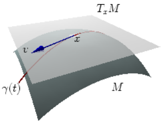

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- gradient of functions

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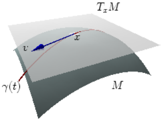

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- angle between curves at a crossing point
- area and volume
- gradient of functions
- divergence of vector fields, ...

M differentiable manifold (Topology and Analysis)

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g Riemannian metric $\rightsquigarrow$ Geometry on M

(distance, angles, area, curvature, ...)

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g shapes M

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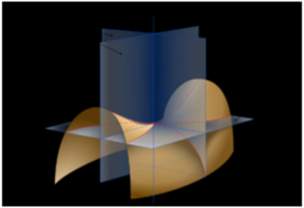
Exercise 1:

WHY? HOW?

Curvature

Curvature



Curvature

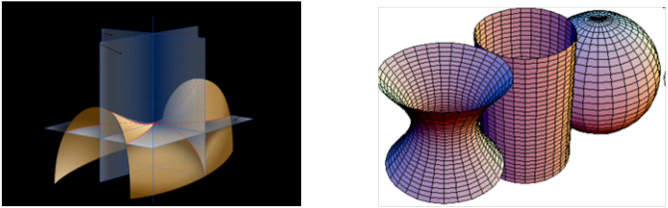
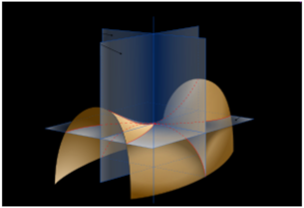
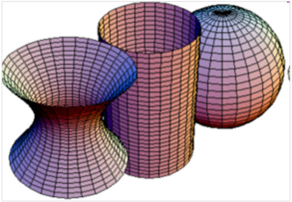


Figure: Gauss curvature

Curvature

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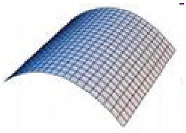
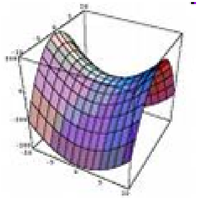

Figure: $K > 0$, $K = 0$, $K < 0$

Sectional curvature

(M, g) Riemannian manifold of dimension n.

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$p \in M, \quad Q \subset \mathrm{T}_p M$ 2-dimensional subspace or plane,

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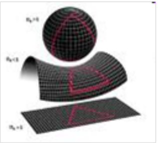

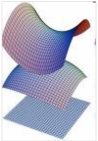

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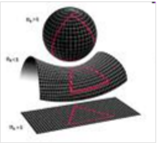

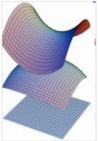

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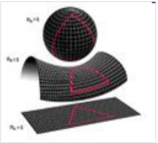

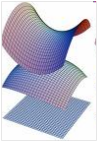

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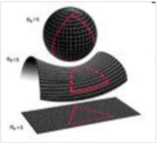

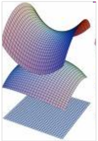

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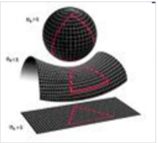

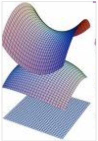

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(4-linear, identities)

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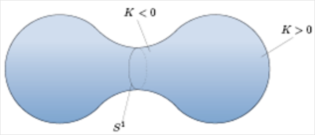

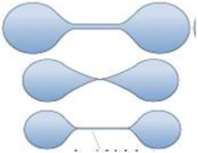

Figure: Las regiones con $K > 0$ tienden a contraerse, y las con $K < 0$ tienden a expandirse.

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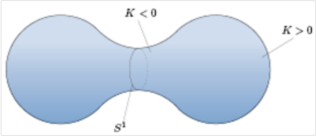

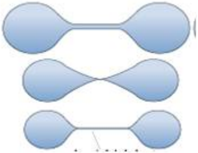

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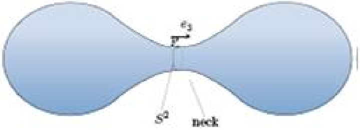

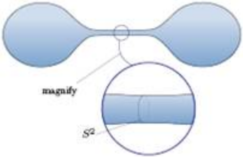

Figure: $\rightsquigarrow$ solitones de Ricci.

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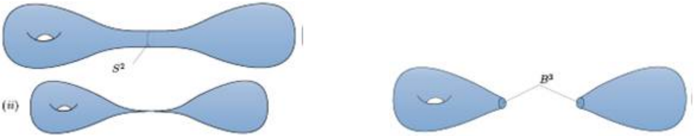

Figure: Cirugía.