The Riesz potential as a multilinear operator into BMO_β spaces

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Contents

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For the case of the fractional integral considered by Kenig and Stein and Grafakos and Torres, [KS, GT], this fact is reflected on the possible values of q in the target space L^q.
Let $n, k \in \mathbb{N}, k \geq 2$,

$$\vec{f} = (f_1, f_2, \dots, f_k) \in L^{p_1} \times \cdots \times L^{p_k},$$

$1 \leq p_i \leq \infty, 1 \leq i \leq k$,
$\vec{y} = (y_1, \dots, y_k) \in (\mathbb{R}^n)^k, x \in \mathbb{R}^n, 0 < \alpha < kn$

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Case $\sum_{i=1}^{k} \frac{1}{p_i} > \frac{\alpha}{n}$

In fact q could be less than one even when all the p_i's are larger than one. This is the case when $\frac{1}{q} = \sum_{i=1}^{k} \frac{1}{p_i} - \frac{\alpha}{n}$ is large.

[KS, GT]$\rightarrow$ $0 < \alpha < kn$, $1 \leq p_i \leq \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}$, $1 \leq i \leq k$

If

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} > 0$$

then

If each $p_i > 1$,

$$\|I_\alpha f\|_{L^q} \leq C \prod_{i=1}^{k} \|f_i\|_{L^{p_i}}$$

$p_i = 1$ for some i,

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In this case we get more regularity in the target space than it could be " linearly" expected.
As it is well known from the linear case, see for instance [SZ, GV, HSV], spaces defined through mean oscillations are expected to appear in the image of the operator.

Spaces BMO_β, $0 \leq \beta < 1$

John and Nirenberg, Spanne, Campanato, [C, JN, S].

Definition

A function $f \in L^1_{loc}(\mathbf{R}^n)$ belongs to BMO_β if for some nonnegative constant C and any ball $B \subseteq \mathbf{R}^n$ of radius R there is a constant a_B such that

$$\frac{1}{|B|}\int_B |f(x) - a_B|dx \leq CR^\beta .$$

This is a Banach space of classes modulo constants, with the norm $\|f\|_\beta$ defined by the infimum of the above constants.

$BMO_0 = BMO$ is the space of functions of bounded mean oscillation.

Redefinition of the operator

An extension of the definition of the fractional integral, as in [SZ, GV, HSV], needed to guarantee the convergence of the integral when $\sum_{i=1}^{k} \frac{1}{p_i}$ is small:

The kernel:

$$K(x, \vec{y}) = (|x - y_1| + |x - y_2| + \cdots + |x - y_k|)^{-(kn-\alpha)}$$

The operator:

$$\mathcal{I}_\alpha^0 \vec{f}(x) = \int \vec{f}(\vec{y}) \left(K(x, \vec{y}) - (1 - \chi_{\mathbf{B}_0}(\vec{y})) K(0, \vec{y}) \right) d\vec{y},$$

$\mathbf{B}_0 = (B_n(0,1))^k$ is the unit cube in $(\mathbb{R}^n)^k$.

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$0 < \alpha < kn$, $1 \leq p_i \leq \infty$, $1 \leq i \leq k$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}$.
If $0 \leq \alpha - \frac{n}{p} < 1$
then $\mathcal{I}_\alpha^0$ is a bounded multilinear operator from $L^{p_1} \times \cdots \times L^{p_k}$ into $BMO_{\alpha - \frac{n}{p}}$.
That is
For some nonnegative constant C, all $\vec{f} \in L^{p_1} \times \cdots \times L^{p_k}$ and any ball B of radius R there is a constant a_B such that

$$\frac{1}{|B|} \int_B |\mathcal{I}_\alpha^0 \vec{f}(x) - a_B| dx \leq C R^{\alpha - \frac{n}{p}} \prod_{i=1}^{k} \|f_i\|_{L^{p_i}}.$$

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$$\mathcal{I}_\alpha^0 : L^{p_1} \times \cdots \times L^{p_k} \to BMO_{\alpha - \frac{n}{p}} \longleftrightarrow P = (\frac{1}{p_1}, \ldots, \frac{1}{p_k})$$

is on the hyperplane of $\mathbb{R}^k$

$$\sum_{i=1}^{k} \frac{1}{p_i} = \frac{1}{p}.$$

Some cases

- If $0 < \alpha < n$ and $\frac{\alpha-1}{n} < \frac{1}{p} \leq \frac{\alpha}{n}$ and $E_1 \dots E_k$ denotes the closed convex hull of the points $E_1 = (\frac{1}{p}, 0, \dots, 0), \dots, E_k = (0, \dots, 0, \frac{1}{p})$. Then

$$\mathcal{I}_\alpha^0 : E_1 \dots E_k \to BMO_{\alpha - n/p}$$

- If $j \leq \frac{1}{p} \leq \frac{\alpha}{n} < j+1$, ($1 \leq j < k$) and $\mathcal{E}$ is the convex hull of the $\binom{k}{j} \times (k-j)$ points whose coordinates are all the permutations of $(\underbrace{1, \dots, 1}_{j}, \frac{1}{p} - j, 0, \dots, 0)$

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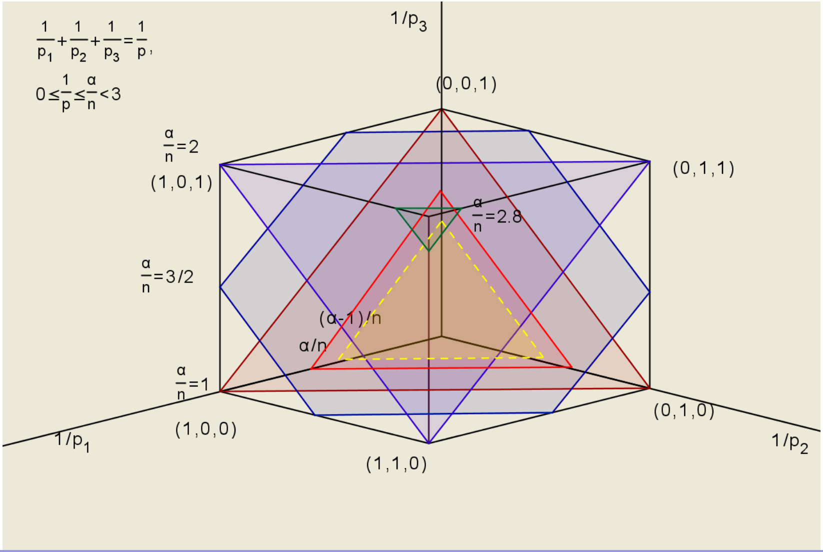
Sketching the Starting Spaces



$\beta = \alpha - \frac{n}{p} \geq 1$?

When P approaches zero, say, $\frac{1}{p} \leq \frac{\alpha-1}{n}$, or $\frac{1}{p} \leq \frac{\alpha-2}{n}$, or smaller, what can be said about the target space?
To answer this question we need first to consider a broader range of spaces of bounded mean oscillation, of classes modulo polynomials, defined in [GC-RdF], gauged by bigger powers of the radius.

More $BM0_\beta$ spaces

$\beta \geq 0$

BMO_β is the space of those locally integrable functions f on $\mathbb{R}^n$, such that for some nonnegative constant C and each ball B in $\mathbb{R}^n$ of radius R there is a polynomial $P_B(f)$, of degree at most the integer part $[\beta]$ of β, such that

$$\frac{1}{|B|}\int_B |f(x) - P_B(f)(x)|dx \leq CR^\beta.$$

The infimum of the above constants will be denoted by $\|f\|_\beta$ and is a norm on BMO_β considered as a space of classes modulo polynomial of order $[\beta]$.

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Extension of the operator: $m \leq \alpha - \frac{n}{p} < m+1$

$m \in \mathbf{N} \cup \{0\}$

We consider the Taylor's polynomial of $K(x,y)$ of degree m with respect to the variable $x \in \mathbb{R}^n$ centered in $(x_0, \vec{y})$.

$$K^m_{x_0}(x,\vec{y}) = \sum_{|\gamma| \leq m} \frac{1}{|\gamma|!} D^\gamma_x K(x_0, y)(x - x_0)^\gamma$$

where $\gamma = (\gamma_1, \ldots \gamma_n) \in \mathbf{N}^n_0$, $|\gamma| = \gamma_1 + \cdots + \gamma_n$, $D^\gamma_x = (\frac{\partial}{\partial x_1})^{\gamma_1} \ldots (\frac{\partial}{\partial x_n})^{\gamma_n}$ and $x^\gamma = x_1^{\gamma_1} \ldots x_k^{\gamma_k}$.

$x_0 = 0$,

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$$\mathcal{I}^m_\alpha \vec{f}(x) = \int_{\mathbb{R}^{nk}} \vec{f}(\vec{y})(K(x,\vec{y}) - (1 - \chi_{\mathbf{B}_0}(\vec{y}))K^m_0(x,\vec{y}))d\vec{y}.$$

This is a natural extension of the definition of the operator given in (1) since :
for f is of compact support included in $B(0,M)$, $I_\alpha\vec{f}(x)$ differs from $\mathcal{I}^m_\alpha\vec{f}(x)$ in a polynomial of degree at most m,

$$\begin{aligned}\mathcal{I}^m_\alpha\vec{f}(x)-I_\alpha\vec{f}(x) &= -\int_{1\leq|y|\leq M}\vec{f}(\vec{y})K^m_0(x,\vec{y})d\vec{y}\\ &= -\sum_{|\gamma|\leq m}\frac{1}{|\gamma|!}x^\gamma\int_{1\leq|y|\leq M}\vec{f}(\vec{y})D^\gamma_x K(0,y)d\vec{y}.\end{aligned}$$

If $\mathcal{I}^m_\alpha\vec{f}(x)$ belongs to BMO_β, $[\beta]=m$, then $I_\alpha\vec{f}(x)$ does.

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Contents

Theorem 2

Let $0 < \alpha < kn$, $1 \leq p_i \leq \infty$ $(1 \leq i \leq k)$ and

$$m \leq \alpha - \frac{n}{p} < m + 1.$$

Then
$\mathcal{I}_\alpha^m$ is a bounded multilinear operator from $L^{p_1} \times \cdots \times L^{p_k}$ into $BMO_{\frac{\alpha}{n} - \frac{1}{p}}$.
That is,
there is a constant C such that for any $f \in L^{p_1} \times \cdots \times L^{p_k}$ and any ball B there is a polynomial of order at most m, $P_B(\mathcal{I}_\alpha^m f)$ such that

$$\frac{1}{|B|} \int_B |\mathcal{I}_\alpha^m f(x) - P_B(\mathcal{I}_\alpha^m f)(x)| dx \leq CR^\beta.$$

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Contents

Notation

$B = B_n(x_0, R) \subseteq \mathbb{R}^n$
$\widetilde{\mathbf{B}} = \{\vec{y} : \|\vec{y} - \widehat{x}_0\| \leq 2kR\} \subseteq (\mathbb{R}^n)^k$
$\|\vec{y}\| = \max(|y_1|, \ldots, |y_k|)$
$\widehat{x} = (x, \ldots, x)$. We will suppose, without loosing generality, that $f_i \geq 0$ for all $1 \leq i \leq k$.

Lemma 1

$0 < \alpha < kn$
$1 \leq p_i \leq \infty$, $1 \leq i \leq k$ $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}$. If

$$\frac{\alpha}{n} - \frac{1}{p} > 0$$

then

$$\frac{1}{|B|} \int_B \int_{\widetilde{\mathbf{B}}} \frac{\vec{f}(\vec{y})}{(\sum_{i=1}^{k} |x - y_i|)^{kn-\alpha}} d\vec{y}\, dx \leq C R^{\alpha - \frac{n}{p}} \prod_{1 \leq i \leq k} \|f_i\|_{p_i},$$

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Denote $\frac{\beta}{n} := \frac{\alpha}{n} - \frac{1}{p} > 0$.

Assume that there is a set of nonnegative numbers $\gamma_1, \ldots, \gamma_k$ such that $\sum_{i=1}^{k} \gamma_i = kn - \alpha$.

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if $1 < p_i \leq \infty$

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$$\gamma_i = \delta \frac{n}{p_i'},\ 1 \leq i \leq k$$

for some $0 < \delta < 1$ (to be determined).

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Lemma 2

Let $0 < \alpha < kn$, $1 \leq p_i \leq \infty$, $(1 \leq i \leq k)$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}$.
If

$$\frac{1}{p} = \frac{\alpha}{n}$$

then

$$\frac{1}{|B|} \int_B \int_{\widetilde{\mathbf{B}}} \frac{\vec{f}(\vec{y})}{\left(\sum_{i=1}^{k} |x - y_i|\right)^{kn-\alpha}} d\vec{y}\, dx \leq C \prod_{1 \leq i \leq k} \|f_i\|_{p_i},$$

for some constant C, all $\vec{f} \in L^{p_1} \times \cdots \times L^{p_k}$, $x \in B = B_n(x_0, R)$.

Proof of Lemma 2

Assume that for some $\gamma_i \geq 0$, $1 \leq i \leq k$ such that $\sum \gamma_i = kn - \alpha$.
Since $0 < \alpha = \frac{n}{p}$ there is at least one i such that $p_i < \infty$. Let assume that $i = 1$ and $p_1 < \infty$. Then $1 < p_1' \leq \infty$ and
we set

$$\gamma_1 = \frac{n}{p_1'} + \epsilon < n$$

and

$$0 < \gamma_i = (1-\delta)\frac{n}{p_i'},\ 2 \leq i \leq k$$

for some $0 \leq \epsilon < \frac{n}{p_1}$ and $0 < \delta < 1$ to be chosen next.
Observe that, for $2 \leq i \leq k$, if $p_i = 1$ then $\gamma_i = 0$ and if $p_i > 1$ then $\gamma_i p_i' = (1-\delta)n < n$.

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Observe that, for $2 \leq i \leq k$, if $p_i = 1$ then $\gamma_i = 0$ and if $p_i > 1$ then $\gamma_i p_i' = (1 - \delta)n < n$.

Moreover,

$$
\begin{aligned}
&\int_{\widetilde{\mathbf{B}}} \frac{\vec{f}(\vec{y})}{(\sum_{i=1}^{k} |x-y_i|)^{kn-\alpha}} d\vec{y} \\
&\quad \leq \prod_{i=1}^{k} \int_{\{y_i : |y_i - x_0| \leq 2kR\}} \frac{f_i(y_i)}{|y_i - x|^{\gamma_i}} dy_i \\
&\quad = \int_{\{y_1 : |y_1 - x_0| \leq 2kR\}} \frac{f_1(y_1)}{|y_1 - x|^{\gamma_1}} dy_1 \prod_{i=2}^{k} \int_{\{y_i : |y_i - x_0| \leq 2kR\}} \frac{f_i(y_i)}{|y_i - x|^{\gamma_i}} dy_i
\end{aligned}
$$

Since $\sum_{i=1}^{k} \frac{n}{p_i'} = kn - \sum_{i=1}^{k} \frac{n}{p_i} = kn - \frac{n}{p} = kn - \alpha = \sum_{i=1}^{k} \gamma_i$ then

$$\sum_{i=2}^{k} (\frac{n}{p_i'} - \gamma_i) = \sum_{i=2}^{k} \frac{n}{p_i'} - \sum_{i=2}^{k} \gamma_i = \gamma_1 - \frac{n}{p_1'}.$$

By Hölder's inequality

$$\prod_{i=2}^{k} \int_{\{y_i : |y_i - x_0| \leq 2kR\}} \frac{f_i(y_i)}{|y_i - x|^{\gamma_i}} dy_i \leq \prod_{i=2}^{k} \int_{\{y_i : |y_i - x| \leq 3kR\}} \frac{f_i(y_i)}{|y_i - x|^{\gamma_i}} dy_i$$

$$\leq C \prod_{i=2}^{k} (\|f_i\|_{p_i} R^{\frac{n}{p_i'} - \gamma_i}) = C \prod_{i=2}^{k} \|f_i\|_{p_i} R^{\gamma_1 - \frac{n}{p_1'}}. \quad (6.1)$$

Since $\sum_{i=1}^{k} \frac{n}{p_i'} = kn - \sum_{i=1}^{k} \frac{n}{p_i} = kn - \frac{n}{p} = kn - \alpha = \sum_{i=1}^{k} \gamma_i$ then

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By Tonelli's Theorem,

$$\frac{1}{|B|}\int_B\int_{\widetilde{\mathbf{B}}}\frac{\vec{f}(\vec{y})}{\left(\sum_{i=1}^k |x-y_i|\right)^{kn-\alpha}}d\vec{y}\,dx$$

$$\begin{aligned}
&\leq C\frac{1}{|B|}R^{\gamma_1-\frac{n}{p_1'}}\prod_{i=2}^k\|f_i\|_{p_i}\int_B\int_{|y_1-x_0|\leq 2kR}\frac{f_1(y_1)}{|x-y_1|^{\gamma_1}}dy_1\,dx\\
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&\leq C\frac{1}{|B|}R^{\gamma_1-\frac{n}{p_1'}}\prod_{i=1}^k\|f_i\|_{p_i}R^{n-\gamma_1+\frac{n}{p_1'}}\\
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\end{aligned}$$

Construction of the γ_i's

Recall that we need

$$\sum_{i=1}^{k} \gamma_i = kn - \alpha > 0$$

$$\gamma_1 = \frac{n}{p_1'} + \epsilon < n$$

and

$$0 \leq \gamma_i = (1-\delta)\frac{n}{p_i'},\ 2 \leq i \leq k$$

for some $0 \leq \epsilon < \frac{n}{p_1}$ and $0 < \delta < 1$.

Since $\frac{n}{p_1'} \leq \sum_{i=1}^{k} \frac{n}{p_i'} = kn - \alpha$ then we can choose $\delta > 0$ near enough to 0 such that

$$0 \leq \epsilon = \delta(kn - \alpha - \frac{n}{p_1'}) < \frac{n}{p_1}.$$

(Observe that if $p_1 = 1$ then $\epsilon = \delta(kn - \alpha) > 0$)

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(Observe that if $p_1 = 1$ then $\epsilon = \delta(kn - \alpha) > 0$)

Since $\sum_{i=2}^{k} \frac{n}{p_i'} = kn - \alpha - \frac{n}{p_1'}$ then

$$\sum_{i=1}^{k} \gamma_i = (\frac{n}{p_1'} + \epsilon) + (1-\delta) \sum_{i=2}^{k} \frac{n}{p_i'}$$

$$= \sum_{i=1}^{k} \frac{n}{p_i'} + \epsilon - \delta \sum_{i=2}^{k} \frac{n}{p_i'} = kn - \alpha + (\epsilon - \delta(kn - \alpha - \frac{n}{p_1'})) = kn - \alpha. \square$$

Lemma 3

Let $0 < \alpha < kn$, $1 \leq p_i \leq \infty$, $(1 \leq i \leq k)$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}$ and $\frac{1}{p} \leq \frac{\alpha}{n}$. For any $m < kn$ nonnegative integer, if

$$m \leq \alpha - \frac{n}{p} < m + 1$$

then

$$\mathcal{I}(x) := \int_{\widetilde{\mathbf{B}}^c} \frac{\vec{f}(\vec{y})}{\left(\sum_{i=1}^{k} |x - y_i|\right)^{kn-\alpha+m+1}} d\vec{y} \leq C R^{\alpha - \frac{n}{p} - m - 1} \prod_{1 \leq i \leq k} \|f_i\|_{p_i}, \qquad (6.2)$$

for some constant C, all $\vec{f} \in L^{p_1} \times \cdots \times L^{p_k}$ and all $x \in B$.

Proof of Lemma 3

Let $\beta = \alpha - \frac{n}{p}$.
If $|x_0 - x| < R$ and $\max_{1 \leq i \leq k} |x_0 - y_i| > 2kR$ then
$|x - y_j| > \frac{1}{2}|x_0 - y_j| = \frac{1}{2}\max_{1 \leq i \leq k} |x_0 - y_i|$ for some j and

$$\sum_{i=1}^{k} |x - y_i| \geq |x - y_j| \geq \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^{k} |x_0 - y_i|\right).$$

Since $kn - \alpha + m + 1 > 0$ then

$$\int_{\widetilde{\mathbf{B}}^c} \frac{\vec{f}(\vec{y})}{\left(\sum_{i=1}^{k} |x - y_i|\right)^{kn-\alpha+m+1}} d\vec{y} \quad \leq \quad C\int_{\widetilde{\mathbf{B}}^c} \frac{\vec{f}(\vec{y})}{\left(\sum_{i=1}^{k} |x_0 - y_i|\right)^{kn-\alpha+m+1}} d\vec{y}$$

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Let $\beta = \alpha - \frac{n}{p}$.
If $|x_0 - x| < R$ and $\max_{1\leq i\leq k} |x_0 - y_i| > 2kR$ then
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Case $m \leq \beta = \alpha < m+1$

Then $p_1 = \cdots = p_k = \infty$ and, since $kn - \alpha + m + 1 > kn$,
then integration on $\mathbf{R}^{nk}$ gives

$$\int_{\widetilde{\mathbf{B}}^c} \frac{\vec{f}(\vec{y})d\vec{y}}{\left(\sum_{i=1}^{k} |x_0 - y_i|\right)^{kn-\alpha+m+1}}$$

$$\leq \quad C \prod_{1\leq i\leq k} \|f_i\|_\infty \int_{\|\vec{y}-\widehat{x}_0\|>2kR} \frac{d\vec{y}}{\left(\sum_{i=1}^{k} |x_0 - y_i|\right)^{kn-\alpha+m+1}}$$

$$\leq \quad CR^{\alpha-m-1} \prod_{1\leq i\leq k} \|f_i\|_\infty \square$$

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Then $p_i < \infty$ for some i.

Aim to write $\sum_{i=1}^{k} |x_0 - y_i|^{kn-\alpha+m+1} \geq |x_0 - y_1|^{\gamma_1} \ldots |x_0 - y_k|^{\gamma_k}$ for some $\gamma_1, \ldots, \gamma_k$ such that $\gamma_i \geq 0$ for all i and $kn - \alpha + m + 1 = \sum_{i=1}^{k} \gamma_i$.

The "cylinders" covering $\widetilde{\mathbf{B}}^c$:

If $\vec{y} \in \widetilde{\mathbf{B}}^c$ there is, at least one, $i \in \{1, \ldots, k\}$ such that $|y_i - x_0| \geq 2kR$.

For any nonempty $A \subseteq I = \{1, \ldots, k\}$, let C_A be the set of all the $\vec{y}$ such that $|y_i - x_0| \geq 2kR$ if $i \in A$, and $|y_i - x_0| \leq 2kR$ if $i \in A^c$.

Then $\widetilde{\mathbf{B}}^c \subseteq \bigcup_{\{A:1\leq \#A \leq k\}} C_A$

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$$\mathcal{I}(x) := \int_{\widetilde{\mathbf{B}}^c} \frac{\vec{f}(\vec{y})}{(\sum_{i=1}^{k} |x - y_i|)^{kn-\alpha+m+1}} d\vec{y}$$
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Let A be any of these subsets. We can assume, without loss of generality, that $A = \{1, \ldots, t\}$ with $1 \leq t \leq k$.

We shall consider the cases $t < k$ and $t = k$.

$$\mathcal{I}(x) := \int_{\widetilde{\mathbf{B}}^c} \frac{\vec{f}(\vec{y})}{\left(\sum_{i=1}^{k}|x-y_i|\right)^{kn-\alpha+m+1}} d\vec{y}$$

$$\leq \sum_{\{A:1\leq \#A\leq k\}} \int_{C_A} \frac{\vec{f}(\vec{y})}{\left(\sum_{i=1}^{k}|x-y_i|\right)^{kn-\alpha+m+1}} d\vec{y}$$

Let A be any of these subsets. We can assume, without loss of generality, that $A=\{1,\ldots,t\}$ with $1\leq t\leq k$.
We shall consider the cases $t<k$ and $t=k$.

$\#A = t < k$

$$\int_{C_A} \frac{\vec{f}(\vec{y})}{\sum_{i=1}^{k} |x_0 - y_i|^{kn-\alpha+m+1}} d\vec{y}$$
$$\leq \prod_{1\leq i\leq t} \int_{\{y_i:|y_i-x_0|\geq 2kR\}} \frac{f_i(y_i)dy_i}{|x_0 - y_i|^{\gamma_i}} \prod_{t+1\leq i\leq k} \int_{\{y_i:|y_i-x_0|\leq 2kR\}} \frac{f_i(y_i)dy_i}{|x_0 - y_i|^{\gamma_i}}$$

For $1 \leq i \leq t$

$$\gamma_i > \frac{n}{p_i'} \text{ if } 1 < p_i \leq \infty$$
$$\gamma_i \geq 0 \text{ if } p_i = 1$$

If so, by Hólder's inequality,

$$\int_{\{y_i:|y_i-x_0|\geq 2kR\}} \frac{f_i(y_i)\,dy_i}{|x_0 - y_i|^{\gamma_i}} \leq C\|f_i\|_{p_i} \frac{1}{R^{\gamma_i - \frac{n}{p_i'}}},$$

$\#A = t < k$

$$\int_{C_A} \frac{\vec{f}(\vec{y})}{\sum_{i=1}^{k} |x_0 - y_i|^{kn-\alpha+m+1}} d\vec{y}$$
$$\leq \prod_{1\leq i\leq t} \int_{\{y_i : |y_i - x_0| \geq 2kR\}} \frac{f_i(y_i) dy_i}{|x_0 - y_i|^{\gamma_i}} \prod_{t+1\leq i\leq k} \int_{\{y_i : |y_i - x_0| \leq 2kR\}} \frac{f_i(y_i) dy_i}{|x_0 - y_i|^{\gamma_i}}$$

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For $t+1 \leq i \leq k$

$$0 \leq \gamma_i < \frac{n}{p_i'} \text{ if } 1 < p_i \leq \infty,$$
$$\gamma_i = 0 \text{ if } p_i = 1.$$

Then

$$\int_{\{y_i : |y_i - x_0| \leq 2kR\}} \frac{f_i(y_i)dy_i}{|x_0 - y_i|^{\gamma_i}} \leq C\|f_i\|_{p_i} R^{\frac{n}{p_i'} - \gamma_i},$$

Asking

$$kn-\alpha+m+1=\sum_{i=1}^{k}\gamma_i,$$

since $\sum_{i=1}^{k}\frac{n}{p_i'}=kn-\frac{n}{p}$ then

$$\sum_{i=1}^{k}\left(\frac{n}{p_i'}-\gamma_i\right)=\left(\alpha-\frac{n}{p}\right)-(m+1)=\beta-(m+1)$$

and

$$\int_{C_A}\frac{\vec{f}(\vec{y})}{\sum_{i=1}^{k}|x_0-y_i|^{kn-\alpha+m+1}}d\vec{y}\leq C\prod_{1\leq i\leq k}\|f_i\|_{p_i}R^{\beta-m-1}.$$

Choosing the γ_i's for $t < k$

Set

$$\gamma_i = \frac{n}{p_i'} + \frac{\epsilon}{t} \text{ if } 1 \leq i \leq t$$
$$\gamma_i = \frac{n}{p_i'}(1-\delta) \text{ if } \ t+1 \leq i \leq k$$

for some $0 < \delta < 1$ and $\epsilon > 0$ to be determined. Then

$$\sum_{i=1}^{k} \gamma_i = \sum_{i=1}^{k} \frac{n}{p_i'} + \epsilon - \delta \sum_{i=t+1}^{k} \frac{n}{p_i'} = kn - \frac{n}{p} + \epsilon - \delta \sum_{i=t+1}^{k} \frac{n}{p_i'} = kn - \alpha + m + 1,$$

if we choose

$$\epsilon - \delta \sum_{i=t+1}^{k} \frac{n}{p_i'} = m + 1 - (\alpha - \frac{n}{p}) = m + 1 - \beta.$$

By letting $\delta \approx 0$ we can choose $\epsilon \approx m + 1 - \beta > 0$.

$A = \{1, \dots, k\}$

$$\gamma_i = \frac{n}{p_i'} + \frac{\epsilon}{k} \text{ for } 1 \le i \le k$$

Then

$$\gamma_i p_i' = n + \frac{\epsilon p_i'}{k} > n \text{ if } 1 < p_i \le \infty$$

$$\gamma_i = \frac{\epsilon}{k} > 0 \text{ if } p_i = 1$$

$$\sum_{i=1}^{k} \gamma_i = \sum_{i=1}^{k} \frac{n}{p_i'} + \epsilon = kn - \frac{n}{p} + \epsilon = kn - \alpha + m + 1$$

if

$$\epsilon = m + 1 - (\alpha - \frac{n}{p}) = m + 1 - \beta > 0.$$

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Then

$$\int_{\{y_i:|y_i-x_0|\geq 2kR\}} \frac{f_i(y_i)dy_i}{|x_0-y_i|^{\gamma_i}} \leq C\|f_i\|_{p_i}\frac{1}{R^{\gamma_i-\frac{n}{p_i'}}} = C\|f_i\|_{p_i}\frac{1}{R^{\frac{\epsilon}{k}}}$$

Thus,

$$\int_{C_A} \frac{\vec{f}(\vec{y})}{\sum_{i=1}^{k}|x_0-y_i|^{kn-\alpha+m+1}}d\vec{y} \leq \prod_{1\leq i\leq k}\int_{\{y_i:|y_i-x_0|\geq 2kR\}}\frac{f_i(y_i)dy_i}{|x_0-y_i|^{\gamma_i}}$$

$$\leq CR^{\beta-1-m}\prod_{1\leq i\leq k}\|f_i\|_{p_i}.\square$$

Then

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Thus,

$$\begin{aligned} \int_{C_A} \frac{\vec{f}(\vec{y})}{\sum_{i=1}^k |x_0-y_i|^{kn-\alpha+m+1}} d\vec{y} &\leq \prod_{1\leq i\leq k} \int_{\{y_i:|y_i-x_0|\geq 2kR\}} \frac{f_i(y_i)dy_i}{|x_0-y_i|^{\gamma_i}} \\ &\leq CR^{\beta-1-m} \prod_{1\leq i\leq k} \|f_i\|_{p_i}. \square \end{aligned}$$

Contents

Proof of Theorem 1

$0 \leq \beta < 1$

To prove:

$$\frac{1}{|B|}\int_B |\mathcal{I}^0_\alpha \vec{f}(x) - a_B|dx \leq CR^{\alpha-\frac{n}{p}} \prod_{i=1}^{k} \|f_i\|_{L^{p_i}}. \qquad (7.3)$$

for some constant a_B, related to $\mathcal{I}^0_\alpha \vec{f}$ and B, and an absolute constant C

Let

$$a_B = \int \vec{f}(\vec{y})\left((1-\chi_{\widetilde{\mathbf{B}}}(\vec{y}))K(x_0,\vec{y}) - (1-\chi_{\mathbf{B}_0}(\vec{y}))K(0,\vec{y})\right)d\vec{y}.$$

Assume that a_B is finite and let us prove (7.3).

Split $\vec{f} = \vec{f}_1 + \vec{f}_2$ where $\vec{f}_1 = \vec{f}\chi_{\widetilde{\mathbf{B}}}$, and

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By Lemma 1 if $\beta := \alpha - \frac{n}{p} > 0$ and Lemma 2 if $\beta = 0$,

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Estimating $\mathcal{I}_2(x)$

Since
$\max_{1\leq i\leq k}|x_0-y_i|>2k|x_0-x|\Rightarrow\sum_{i=1}^k|x-y_i|>k|x_0-x|$ then
using the Mean Value Theorem and the homogeneity of order $\alpha-kn$ of the kernel $K(x,\vec{y})$,

$$|K(x,\vec{y})-K(x_0,\vec{y})|\leq C\frac{|x-x_0|}{(\sum_{i=1}^k|x_0-y_i|)^{kn-\alpha+1}},$$

For $x\in B$ and $\vec{y}\in\widetilde{\mathbf{B}}^c$ and Lemma 3 ($0\leq\beta<1orm=0$),

$$\mathcal{I}_2(x)\leq C|x-x_0|\int_{\widetilde{\mathbf{B}}^c}\frac{\vec{f}(\vec{y})}{(\sum_{i=1}^k|x_0-y_i|)^{kn-\alpha+1}}d\vec{y}$$
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Estimating $\mathcal{I}_2(x)$

Since
$\max_{1\leq i\leq k}|x_0-y_i| > 2k|x_0-x| \Rightarrow \sum_{i=1}^{k}|x-y_i| > k|x_0-x|$ then
using the Mean Value Theorem and the homogeneity of order $\alpha - kn$ of the kernel $K(x,\vec{y})$,

$$|K(x,\vec{y}) - K(x_0,\vec{y})| \leq C\frac{|x-x_0|}{(\sum_{i=1}^{k}|x_0-y_i|)^{kn-\alpha+1}},$$

For $x \in B$ and $\vec{y} \in \widetilde{\mathbf{B}}^c$ and Lemma 3 $(0 \leq \beta < 1 orm = 0)$,

$$\mathcal{I}_2(x) \leq C|x-x_0| \int_{\widetilde{\mathbf{B}}^c} \frac{\vec{f}(\vec{y})}{(\sum_{i=1}^{k}|x_0-y_i|)^{kn-\alpha+1}} d\vec{y}$$
$$\leq CR \times R^{\beta-1} \prod_{1\leq i\leq k} \|f_i\|_{p_i} = CR^{\beta} \prod_{1\leq i\leq k} \|f_i\|_{p_i}$$

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a_B is finite: easy $\square$

Proof of Theorem 2

$m = [\beta]$.

$m = 0 \to$ Theorem 1.

$m \geq 1$:

For $x \in B(x_0, R) \subset \mathbf{R}^n$ let

$$a_B(x) = \int_{\mathbb{R}^{nk}} \vec{f}(\vec{y}) \left((1 - \chi_{\widetilde{\mathbf{B}}}(\vec{y})) K^m_{x_0}(x, \vec{y}) - (1 - \chi_{\mathbf{B}_0}(\vec{y})) K^m_0(x, \vec{y}) \right) d\vec{y},$$

$\vec{f} = \vec{f}_1 + \vec{f}_2 \quad \vec{f}_1 = \vec{f} \chi_{\widetilde{\mathbf{B}}}$

$$\begin{aligned}
&\mathcal{I}^m_\alpha \vec{f}(x) - a_B(x) \\
&= \int_{\mathbb{R}^{nk}} \vec{f}(\vec{y}) \left(K(x, \vec{y}) - (1 - \chi_{\widetilde{\mathbf{B}}}(\vec{y})) K^m_{x_0}(x, \vec{y}) \right) d\vec{y} \\
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Proof of Theorem 2

$m = [\beta]$.

$m = 0 \rightarrow$ Theorem 1.

$m \geq 1$:

For $x \in B(x_0, R) \subset \mathbf{R}^n$ let

$$a_B(x) = \int_{\mathbb{R}^{nk}} \vec{f}(\vec{y}) \left((1 - \chi_{\widetilde{\mathbf{B}}}(\vec{y})) K^m_{x_0}(x, \vec{y}) - (1 - \chi_{\mathbf{B}_0}(\vec{y})) K^m_0(x, \vec{y}) \right) d\vec{y},$$

$\vec{f} = \vec{f}_1 + \vec{f}_2 \quad \vec{f}_1 = \vec{f} \chi_{\widetilde{\mathbf{B}}}$

$$\begin{aligned} &\mathcal{I}^m_\alpha \vec{f}(x) - a_B(x) \\ &= \int_{\mathbb{R}^{nk}} \vec{f}(\vec{y}) \left(K(x, \vec{y}) - (1 - \chi_{\widetilde{\mathbf{B}}}(\vec{y})) K^m_{x_0}(x, \vec{y}) \right) d\vec{y} \\ &= \int \vec{f}_1(\vec{y}) K(x, \vec{y}) d\vec{y} + \int \vec{f}_2(\vec{y}) \left(K(x, \vec{y}) - K^m_{x_0}(x, \vec{y}) \right) d\vec{y} \\ &= \mathcal{I}_1(x) + \mathcal{I}_2(x), \end{aligned}$$

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$\beta > 0 \to$ Lemma 1$\to$

$$\frac{1}{|B|}\int_B |\mathcal{I}_1(x)|\,dx \le CR^{\beta} \prod_{1\le i\le k} \|f_i\|_{p_i}. \tag{7.6}$$

$\max_{1\le i\le k}|x_0 - y_i| > 2k|x - x_0|$ and Taylor's theorem:

$$\begin{aligned}
|K(x,\vec{y}) - K^m_{x_0}(x,\vec{y})| &= |K(x,\vec{y}) - \sum_{|\gamma|\le m}\frac{1}{|\gamma|!}D^{\gamma}_x K(x_0,y)(x-x_0)^{\gamma}| \\
&\le C \max_{|\xi - x_0|\le|x-x_0|,|\gamma|=m+1} |D^{\gamma}_x K(\xi,\vec{y})|\,|x-x_0|^{m+1} \\
&\le C\frac{|x-x_0|^{m+1}}{(\sum_{i=1}^k |x_0-y_i|)^{kn-\alpha+m+1}}.
\end{aligned}$$

By Lemma 3,

$$\frac{1}{|B|}\int_B |\mathcal{I}_2(x)|\,dx \le C(k,n,\alpha)R^{m+1}\int_{\mathbf{B}^c}\frac{\vec{f}(\vec{y})d\vec{y}}{(\sum_{i=1}^k |x_0-y_i|)^{kn-\alpha+m+1}} \le CR^{\beta}\prod_{1\le i\le k}\|f_i\|_{p_i}. \tag{7.7}$$

$\beta > 0 \rightarrow$ Lemma 1$\rightarrow$

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(7.7) and (7.6) $\rightarrow$

$$\frac{1}{|B|}\int_B |\mathcal{I}_\alpha \vec{f}(x) - a_B(x)|\, dx \leq CR^\beta \prod_{1\leq i\leq k} \|f_i\|_{p_i}.$$

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BUSCADOS!!!