Regresión no paramétrica para datos funcionales

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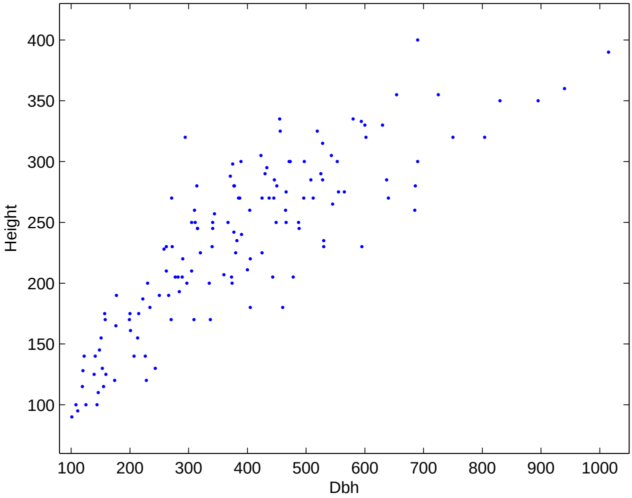
IMAL 2010

The data file ufcgf.txt gives the diameter Dbh in millimeters at 137 cm perpendicular to the bole, and the Height of the tree in decimeters for a sample of grand fir trees at Upper Flat Creek, Idaho, in 1991, courtesy of Andrew Robinson.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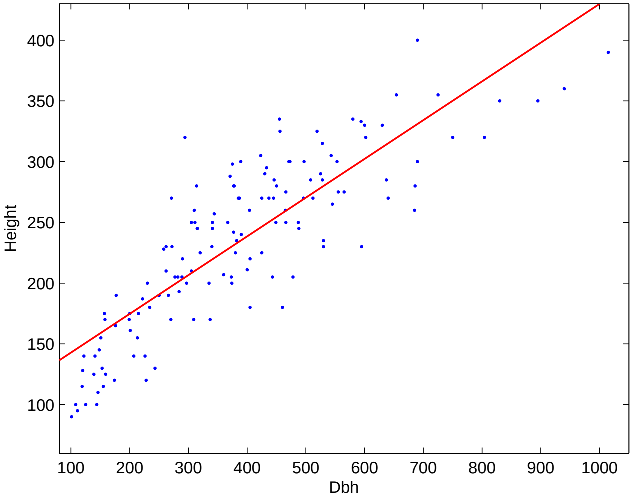
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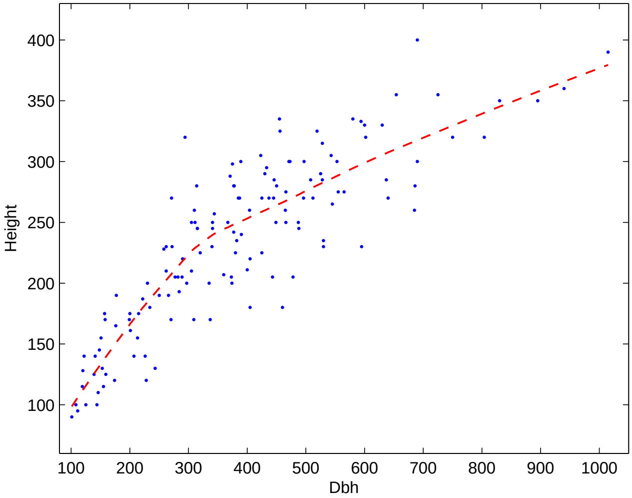
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Regresión en $\mathbb{R}^d$

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Estimar el operador de regresión $\eta : \mathbb{R}^d \to \mathbb{R}$ en el modelo

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$$(X_i, Y_i)_{i=1}^n \in \mathbb{R}^d \times \mathbb{R}, \text{ i.i.d. } \leftrightsquigarrow (X, Y)$$

$$\widehat{\eta}(X) \doteq \sum_{i=1}^n W_{ni}(X) Y_i.$$

Teorema (Stone 1977)

(i) *Existe $c > 0$ tal que, para toda f, $\mathbb{E}(f(X)) < \infty$,*

$$\mathbb{E}\left(\sum_{i=1}^{n} W_{ni}(X) f(X_i)\right) \leq c\mathbb{E}(f(X)),$$

(ii)

$$\lim_{n\to\infty} \mathbb{E}\left(\sum_{i=1}^{n} W_{ni}(X)\mathbb{I}_{\{\|X_i - X\| > a\}}\right) = 0 \ \forall a > 0,$$

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$$\lim_{n\to\infty} \mathbb{E}\left(\max_{1\leq i\leq n} W_{ni}(X)\right) = 0.$$

Entonces,

$$\lim_{n\to\infty} \mathbb{E}\left((\eta(X) - \widehat{\eta}(X))^2\right) = 0.$$

Ejemplo: Estimador de k-vecinos más cercanos.

$$\widehat{\eta}(X) = \sum_{i=1}^{n} W_{ni}(X) Y_i,$$

$$W_{ni}(X) = \begin{cases} 1/k & \text{si } \|X_i - X\| \leq \left\|X_{(k)} - X\right\|, \\ 0 & \text{en otro caso.} \end{cases}$$

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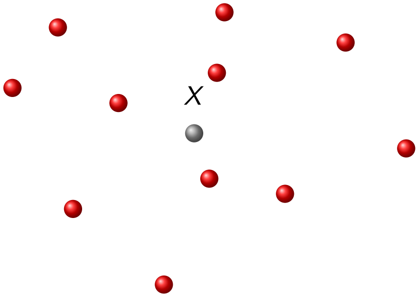

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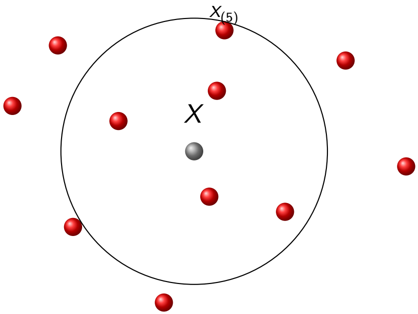

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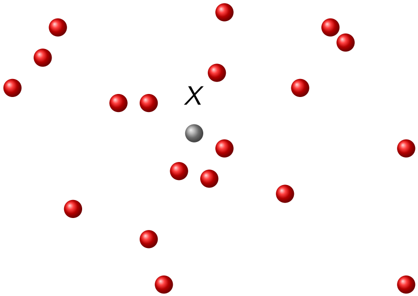

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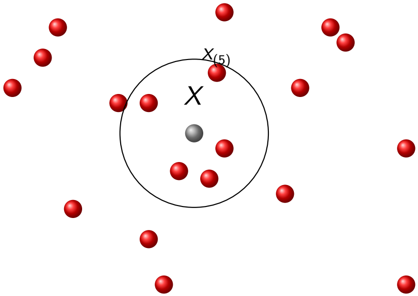

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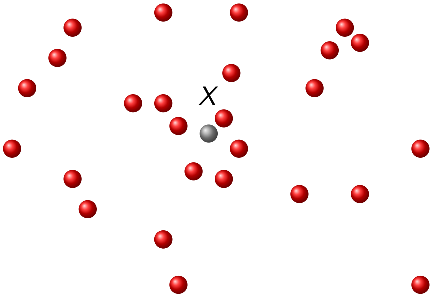

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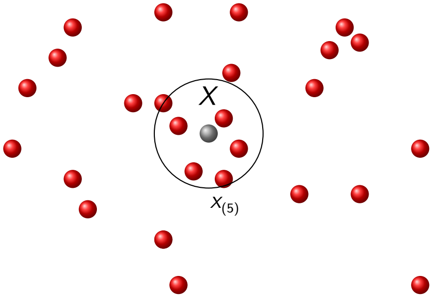

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(iii) *η puede aproximarse en L^2 por funciones uniformemente continuas y acotadas.*

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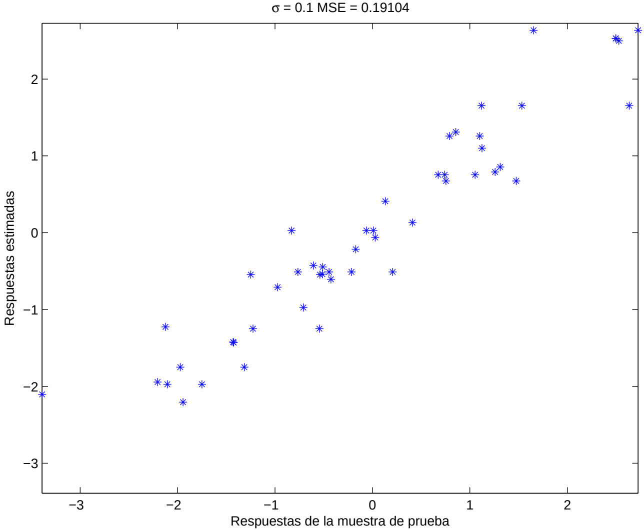
Simulación

$$Y = \eta(\mathcal{X}) + e,$$

- $e \sim \mathcal{N}(0, \sigma)$
- $\eta(\mathcal{X}) = \int_0^1 \mathcal{X}(t)dt,$
- $\mathcal{X}(t) = W(t)$
  - $W(0) = 0,$
  - $W(t) - W(s) \sim \mathcal{N}(0, t - s)$ para $0 \leq s \leq t.$


σ = 0.1 MSE = 0.19104
Respuestas estimadas
Respuestas de la muestra de prueba
−3
−2
−1
0
1
2

Estimador basado en la medida de ocupación

Estimador basado en la medida de ocupación

$$(\mathcal{X}_i, Y_i)_{i=1}^n \in E \times \mathbb{R}, \text{ i.i.d.} \leftrightsquigarrow (\mathcal{X}, Y)$$

Estimador basado en la medida de ocupación

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Definimos el estimador del operador de regresión η

$$\widehat{\eta}(\mathcal{X}) \doteq \sum_{i=1}^n W_{ni}(\mathcal{X}) Y_i,$$

$$W_{ni}(\mathcal{X}) = W_{ni}(\mathcal{X}, \mathcal{X}_1, \ldots, \mathcal{X}_n) = \frac{1}{k_n} \int_T \mathbb{I}_{\{t : |\mathcal{X}(t) - \mathcal{X}_i(t)| \leq H_{n,\mathcal{X}}\}} \, dt.$$

Estimador basado en la medida de ocupación

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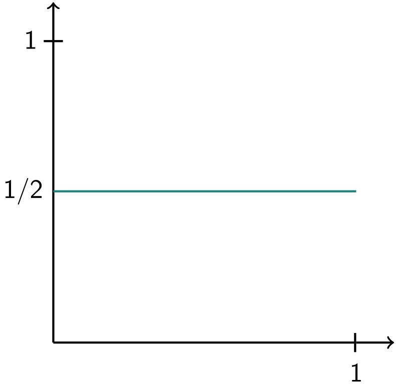
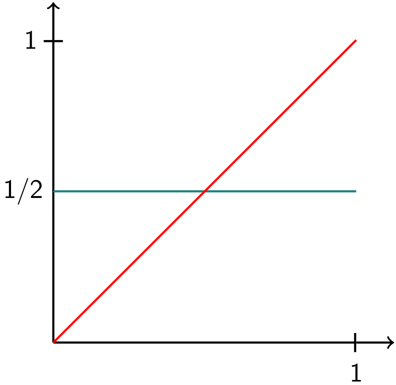
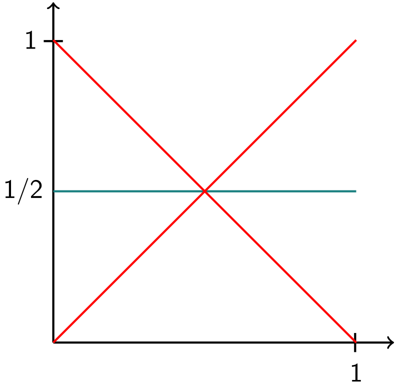
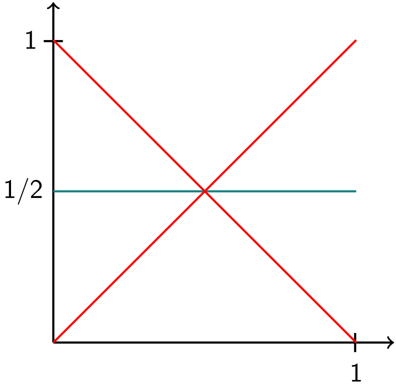
$$k_n \rightsquigarrow H_{n,\mathcal{X}}, \quad k_n = \sum_{i=1}^n \int_T \mathbb{I}_{\{t : |\mathcal{X}(t) - \mathcal{X}_i(t)| \leq H_{n,\mathcal{X}}\}} \, dt.$$


1
1/2
1

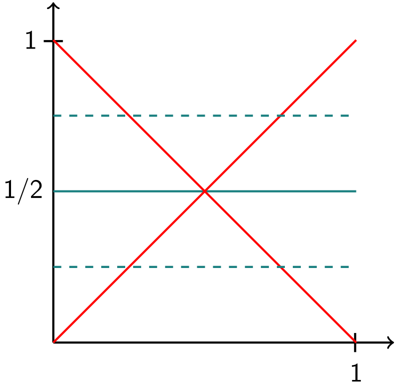

1
1/2
1


1
1/2
1

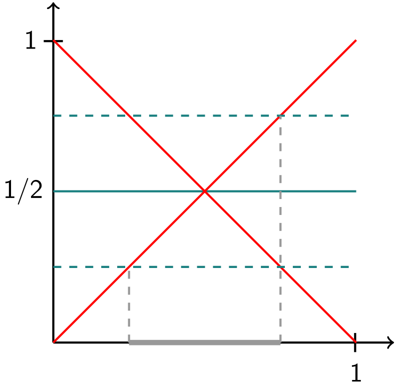
$$\text{Si } k_2 = 1, \text{ ¿} H_{2,\mathcal{X}}, \sum_{i=1}^{2} \int_T \mathbb{I}_{\{t : |\mathcal{X}(t) - \mathcal{X}_i(t)| \leq H_{2,\mathcal{X}}\}} \, dt = 1?$$



$$\text{Si } k_2 = 1,\ ¿H_{2,\mathcal{X}},\ \sum_{i=1}^{2} \int_T \mathbb{I}_{\{t: |\mathcal{X}(t) - \mathcal{X}_i(t)| \le H_{2,\mathcal{X}}\}}\, dt = 1?$$

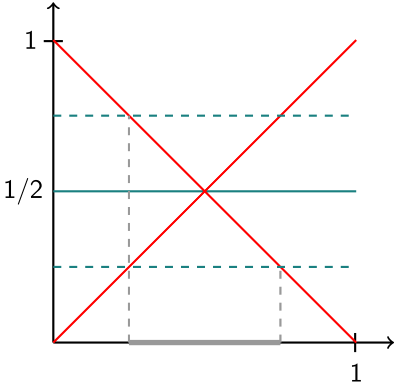


$$\text{Si } k_2 = 1,\ ¿H_{2,\mathcal{X}},\ \sum_{i=1}^{2} \int_T \mathbb{I}_{\{t : |\mathcal{X}(t) - \mathcal{X}_i(t)| \leq H_{2,\mathcal{X}}\}}\, dt = 1?$$



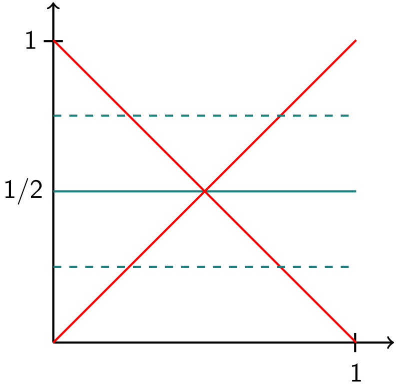
$$\frac{1}{2}$$

$$\text{Si } k_2 = 1, \text{ ¿} H_{2,\mathcal{X}}, \sum_{i=1}^{2} \int_T \mathbb{I}_{\{t : |\mathcal{X}(t) - \mathcal{X}_i(t)| \leq H_{2,\mathcal{X}}\}} \, dt = 1?$$



$$\frac{1}{2} + \frac{1}{2} = 1$$

$$\text{Si } k_2 = 1,\ ¿H_{2,\mathcal{X}},\ \sum_{i=1}^{2} \int_T \mathbb{I}_{\{t:|\mathcal{X}(t)-\mathcal{X}_i(t)|\leq H_{2,\mathcal{X}}\}}\, dt = 1?$$



Respuesta: $H_{2,\mathcal{X}} = \frac{1}{4}$.

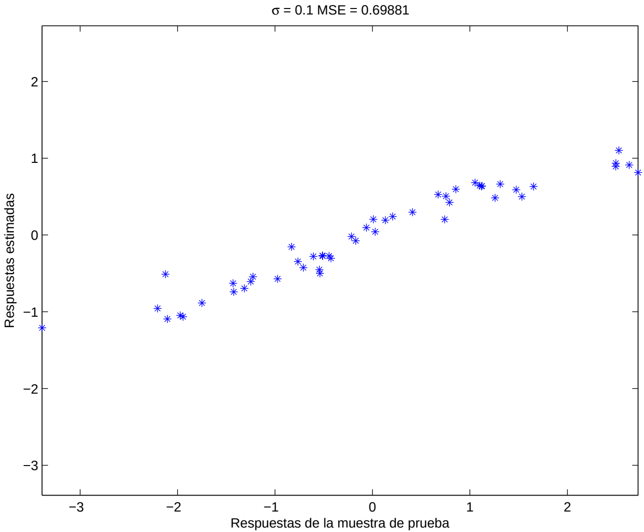
$$¿\lim_{n\to\infty} \mathbb{E}\left((\widehat{\eta}(\mathcal{X}) - \eta(\mathcal{X}))^2\right) = 0?$$

Simulación

$$Y = \eta(\mathcal{X}) + e,$$

- $e \sim \mathcal{N}(0, \sigma)$
- $\eta(\mathcal{X}) = \int_0^1 \mathcal{X}(t)dt,$
- $\mathcal{X}(t) = W(t)$
  - $W(0) = 0,$
  - $W(t) - W(s) \sim \mathcal{N}(0, t-s)$ para $0 \leq s \leq t.$

Simulación


σ = 0.1 MSE = 0.69881
Respuestas estimadas
Respuestas de la muestra de prueba
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2

Gracias!