The logic of rational polyhedra

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A polyhedron is a finite union $P$ of simplexes $S_i$ in $\mathbb{R}^n$.

- $P$ need not be convex.
- $P$ need not be connected.
- $P$ may have parts of different dimensions.

A polyhedron $P = \bigcup S_i$ is said to be **rational** if so are the vertices of every simplex $S_i$. 
Erlangen geometry of a group of transformations

Every group \( G \) of transformations in \( \mathbb{R}^n \) generates a geometry.

Example: \( E_n = \text{the euclidean group of affinities in } \mathbb{R}^n \)
A typical element of \( E_n \) is a map of the form \( x \rightarrow O_n x + t \)
where \( O \) is an orthogonal \( nxn \) matrix, and \( t \) is in \( \mathbb{R}^n \)

\( E_n \) is the semidirect product of the orthogonal group \( O_n \) and \( \mathbb{R}^n \)

We are all familiar with \( E_n \)-invariant measures:
Lebesgue measure is invariant under the euclidean group $E_n$. As a consequence of additivity, Lebesgue measure is invariant under piecewise linear 1-1 maps $h$, provided each linear piece of $h$ is given by some element of $E_n$. 
for each $n=1,2,...$, let us consider the geometry arising from the group $G_n$ of affine maps in $\mathbb{R}^n$ of the form

$$x \mapsto Ux + t$$

where $U$ is an integer $n \times n$ matrix with determinant $\pm 1$, and $t$ is an integer vector in $\mathbb{Z}^n$.

$G_n$ is known as the semidirect product of the unimodular group $GL(n, \mathbb{Z}) = \text{aut}(\mathbb{Z}^n)$ and the translation group $\mathbb{Z}^n$.

we will construct $G_n$-invariant measures. By additivity, these are automatically invariant under piecewise linear 1-1 maps where each piece belongs to $G_n$. 
Z-homeomorphism = PL-homeomorphism with integer coefficients

DEFINITION Two rational polyhedra $P$ and $Q$ are **Z-homeomorphic** if there is a PL-homeomorphism $h$ of $P$ onto $Q$ such that every piece of $h$ as well as of its inverse $h^{-1}$ has integer coefficients.
the action of piecewise $G_n$-linear maps

before

after
motivation: why logic?

rational polyhedra are the affine varieties of the “polynomials” given by formulas in a certain logic $L_\infty$.

A $\mathbb{Z}$-homeomorphism corresponds to isomorphism in the algebras of $L_\infty$, just as homeomorphism corresponds to isomorphism in the algebras of boolean logic (by Stone duality theorem).
\(\mathbb{Z}\)-homeomorphism does not preserve the usual measure of rational polyhedra \(P\) in \(\mathbb{R}^n\) when \(\dim(P) < n\).

These two black segments are \(\mathbb{Z}\)-homeomorphic, but their lengths are different.
to construct an invariant measure of rational polyhedra in the geometry of the group $G_n$
we need the following fundamental notion (taken from algebraic geometry)
DEFINITION The denominator $d = \text{den}(x)$ of a rational point $x$ is the least common denominator of the coordinates of $x$.

DEFINITION A simplex $T$ is regular if it is rational, and for each face $F$ of $T$, each rational point in the interior of $F$ has a denominator $\geq$ the sum of the denominators of the vertices of $F$. 

regularity of a simplex $T = \text{conv}(v_0, \ldots, v_n)$
regular triangulation of a rational polyhedron

all its simplexes are regular

Note: each vertex \((x/d, y/d)\) is written in homogeneous form, \((x, y, d)\)

**Minkowski proved:** The regularity of a simplex \(T\) means that the matrix of the homogeneous coordinates of the vertices of \(T\) is (extendible to) a unimodular integer matrix
volume of a regular simplex $T = \text{conv}(v_0, \ldots, v_n)$

$$\text{vol}(T) = \frac{1}{(n! \text{den}(v_0) \cdots \text{den}(v_n))^{-1}}$$

1/(2x1x1x2)=1/4= the Lebesgue area of the triangle

1/(2x3x2x2)=1/24 also coinciding with its Lebesgue area...
the volume of an arbitrary rational polyhedron $P$ (equipped with a regular triangulation $\Delta$)

We first calculate the volume of each simplex $d\Delta$ of maximum dimension.

Then we set

$$\text{Vol}(P) = \sum \text{Vol}(d\Delta)$$

to show that all this makes mathematical sense, we need a couple of results from toric varieties
dynamics of regular triangulations

unstarring at [a,b,c]
starring at e

unstarring at [a,b]
starring at u
a first main result
(the solution of the weak Oda conjecture by Wlodarczyk-Morelli)

THEOREM Any two regular triangulations of the same rational polyhedron are connected by a path of blow-ups and blow-downs.
a second main result
(elimination of points of indeterminacy in toric varieties)

THEOREM (de Concini-Procesi)  For any two regular triangulations $\Delta$ and $\Sigma$ on the same rational polyhedron, a sequence of blow-ups leads from $\Delta$ to a subdivision of $\Sigma$
by successive blowing ups, we will be able to refine any rational triangulation
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why toric varieties?
given a rational point \( A = (x_1, \ldots, x_n) \) in \( \mathbb{R}^n \)

let \( d \) be the denominator of \( A \)

then the tuple \( d(x_1, \ldots, x_n, 1) \) is a vector \( A' \) in \( \mathbb{Z}^{n+1} \)

\( A' \) is called the homogeneous correspondent of \( A \).
rational simplex $\leftrightarrow$ integral cone

RATIONAL POINT

(PRIMITIVE) INTEGER VECTOR

2-SIMPLEX
\(\text{conv}(A, B, C)\)

3-CONE
\(\langle A', B', C' \rangle = \)
rational triangulation $\iff$ fan

A simplicial complex with rational vertices in $\mathbb{Q}^2$

Any two simplices intersect in a common (possibly empty) face

Its corresponding fan,

A complex of cones with integer vectors
regular triangulation<-->regular fan

passing to homogeneous integer coordinates, every regular (unimodular) triangulation determines

a regular (nonsingular, smooth) fan, a standard tool in algebraic geometry to code nonsingular toric varieties
THEOREM \( \sum \text{Vol}(d\Delta) \) does not depend on \( \Delta \). So the notation \( \text{Vol}(P) \) is unambiguous.

This follows from the proof of Oda's conjecture, upon noting that \( \text{Vol}(P) \) is invariant under blow-ups.
THEOREM  If P and Q are \( \mathbb{Z} \)-homeomorphic rational polyhedra then \( \text{Vol}(P) = \text{Vol}(Q) \)

by the De Concini-Procesi theorem, given \( h \) we can always compute the volumes of \( P \) and \( Q \) with the help of a regular triangulation \( \Delta \) of \( P \) such that \( h \) is linear over each simplex of \( \Delta \).
THEOREM  When P is full-dimensional, 
\[ \sum \text{Vol}(d\Delta) \] is the Lebesgue measure of P

THEOREM  When P is Lebesgue-negligible (as a lower-dimensional polyhedron) still, \[ \sum \text{Vol}(d\Delta) \] is nonzero
The regular triangulation $\Delta$ to compute the integral $\int_P f \, d\Delta$ is so chosen that $f$ is linear on each simplex of $\Delta$

The integral of $f$ over $P$ is now defined in the natural way, as the volume underlying the graph of $f$
We have thus attached to every rational polyhedron $P$ a measure that is invariant under $\mathbb{Z}$-homeomorphisms, coincides with Lebesgue measure if $P$ is full-dimensional, but does not vanish if $P$ is lower-dimensional.
connections with logic
(an introduction for non-logicians)
a main merit of classical logic

to give a rigorous meaning to the statement

conclusion \( p \) “follows” from premises \( p_1, \ldots, p_n \)

“consequence” becomes a mathematical notion
A main merit of $L_\infty$ is to give a rigorous meaning to the following statement:

$p$ “stably” follows from premises $p_1, \ldots, p_n$

in the sense that, even if we randomly delete a certain percentage of the formulas, formula $p$ still follows (in the sense of the previous slide) from the remaining formulas $p_1, \ldots, p_n$.

$L_\infty$ is a mathematically interesting logic for the treatment of partially unreliable information. $L_\infty$ is the logic of the (Rényi-Ulam) Twenty Questions game, where a certain number of answers may be distorted/wrong/mendacious.
basic reference on Lukasiewicz logic $L_\infty$

- any formula $F$ in $L_\infty$ describes the output of a continuous spectrum observable or event, just as a formula in classical boolean logic describes a yes-no event

- $\text{Mod}(F)$, the set of models of $F$, is the most general rational polyhedron

- $\text{Mod}(T)$, for $T$ a set of formulas, is the most general compact Hausdorff space
the $L_\infty$ language

- incorporates numbers and percentages in the language, without mentioning them

- we too, in everyday life, do not quantify our dubiousness degrees when reasoning informally

- rather we prefer to use adjectives or adverbs, like “uncertain” or “moderately unreliable”—and we are still able to make reasonable inferences

- only in classical logic and mathematical reasoning we assume 100% reliability
formulas in one variable
### a formula $f$ in two variables

- $f$ is continuous
- $f$ has finitely many linear pieces
- each piece of $f$ has the form $a_1x_1 + \ldots + a_nx_n + b$
- where $b$ and the $a$’s are integers.
- Any function $f$ with these properties is called a McNaughton function.
the domain of $f$ can be decomposed into finitely many simplexes $S_i$ in such a way that $f$ is linear over each $S_i$.

its zeroset
$Z(f) = f^{-1}(0)$
polyhedra as “affine varieties” of formulas

$L_\infty$-formulas determine the most general possible rational polyhedron in $[0,1]^n$

**rational polyhedra = “affine varieties” of $L_\infty$-formulas**

a formula $F$ in $L_\infty$ and its set of models $\text{Mod}(F) = F^{-1}(1) = \text{set of truth-valuations that satisfy } F$
MV-algebras are the algebras of $L_\infty$-formulas

These are the defining equations of MV-algebras

$\quad x \oplus (y \oplus z) = (x \oplus y) \oplus z$

$\quad x \oplus y = y \oplus x$

$\quad x \oplus 0 = x$

$\quad \neg \neg x = x$

$\quad x \oplus \neg 0 = \neg 0$

Boolean algebras stand to classical logic as MV-algebras stand to $L_\infty$.

Boolean algebras are obtained by adjoining the equation $x+x=x$

$\quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. 
**COROLLARY** Given a rational polyhedron $P$ in the $n$-cube, let $J(P)$ be the set of McNaughton functions of the free MV-algebra $\text{FREE}_n$ vanishing over $P$. Then $J(P)$ is a principal ideal of the free algebra $\text{FREE}_n$.

Conversely, for every principal ideal $J$ of $\text{FREE}_n$ let $Z(J)$ be the intersection of the zerosets of all functions in $J$. Then $Z(J)$ is a polyhedron in the $n$-cube, which coincides with the zeroset of any generator $j$ of $J$.

The two maps $P \rightarrow J(P)$ and $J \rightarrow Z(J)$ are mutually inverse of each other.

These two maps induce a one-one correspondence between rational polyhedra and finitely presented MV-algebras.
closing a circle of ideas: invariant measures on polyhedra are in 1-1 correspondence with invariant probability measures on formulas
• a **state** $f$ of $A$ is a normalized functional on $A$ which is additive on incompatible elements of $A$

• **THEOREM** (Kroupa-Panti) The states of any MV-algebra $A$ are in one-one correspondence with the regular Borel probability measures on the maximal space $\mu(A)$ of $A$

• thus the finitely additive algebraic notion of state corresponds to the usual notion of sigma-additive regular Borel probability
the ratio $\int_P f \, d\Delta / \int_P d\Delta$ is a *computable* rational number, once the function $f$ is presented via a formula of Lukasiewicz logic

this ratio does not depend on the regular triangulation $\Delta$

the map $f \mapsto \int_P f \, d\Delta / \int_P d\Delta$ is an *invariant* state of the finitely presented MV-algebra $A(P)$ corresponding to $P$, called the *Lebesgue state* of $A(P)$, and denoted $L_{A(P)}$

a state $f$ is *invariant* if $f(a(x)) = f(x)$ for every $x$ in $A(P)$ and automorphism $a$ of $A(P)$
• let Q be a variable rational polyhedron in some cube $[0,1]^n$. This $Q$ is the model-set of a formula $G$ in Lukasiewicz logic.

• given any other formula $F$ with its McNaughton function $f_F$, the integral of $f_F$ over $Q$, divided by $\text{Vol}(Q)$ is a conditional probability $P(F|G)$ of $F$ given $G$

• $P(F|G)$ has various properties: rationality, computability, invariance, substitutability: $P(B|C) = P(X|(C\&X \Leftrightarrow B))$

• and also satisfies Rényi’s “law of compound probabilities”, which for yes-no events reads:

• $P(A\&B|C) = P(A|B\&C) \cdot P(B|C)$
this talk was only aimed at showing that the notion of \(Z\)-homeomorphism is dual to the notion of MV-algebraic isomorphism, and thus comes from Lukasiewicz logic

\(Z\)-homeomorphism is at the very beginning of an extensive and deep theory, involving fans, ordered groups, abstract simplicial complexes, probability theory and \(C^*\)-algebras
MV-algebras and their states inside mathematics

- Countable MV-algebras correspond to those AF C*-algebras whose Murray-von Neumann order of projections is a lattice.
- Invariant states of MV-algebras yield invariant states on their corresponding AF C*-algebras.

