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**GRADUAL DOUBLING PROPERTY OF HUTCHINSON ORBITS**

By

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## GRADUAL DOUBLING PROPERTY OF HUTCHINSON ORBITS

HUGO AIMAR, MARILINA CARENA, AND BIBIANA IAFFEI

ABSTRACT. The classical self-similar fractals can be obtained as fixed points of the iteration technique introduced by Hutchinson. The well known results of Mosco show that typically the limit fractal equipped with the invariant measure is a (normal) space of homogeneous type. But the doubling property along this iteration is generally not preserved even when the starting point, and of course the limit point, have both the doubling property. We prove that the elements of Hutchinson orbits satisfy the doubling property except perhaps for radii which decrease to zero as the step of the iteration grows, and in this sense, we say that the doubling property of the limit is achieved gradually. We use this result to prove the uniform upper doubling property of the orbits.

## 1. INTRODUCTION

Since the earlier works by Coifman and Guzmán in [5] and Coifman and Weiss in [6], the harmonic analysis has a natural environment the (quasi) metric spaces with doubling measures. There the expression *space of homogeneous type* for this setting seems to be coined. Since the doubling property allows the use of Wiener type covering lemmas, many of the basic results of harmonic analysis have been extended to the setting of space of homogeneous type.

On the other hand, the results in [17] show that typically the limit fractal provided by the Hutchinson iteration scheme (see [12]) and equipped with the invariant measure, is a (normal) space of homogeneous type. Some current attempts to extend notions of harmonic analysis and partial differential equations to fractals (see [18], [14], [15]) suggest that some results of real and functional analysis on these settings are of interest.

Since the classical self-similar fractals are actually obtained as fixed points of the iteration technique introduced by Hutchinson, one may ask for the preservation of the doubling property along the iteration. If this property is uniformly preserved, one can expect that the behavior of some operators on the limit fractal could be predicted from approximate versions defined on the simpler approximation spaces (see [1]). But the doubling property is generally not preserved by the iteration procedure. In [2] we prove that it may happen that no point, except for the first and last, of the orbit generated by a contraction is a space of homogeneous type, even when the starting point, and of course the limit point, have both the doubling property. On the other hand, the uniqueness of the (Banach) fixed point lead us to the same limit space no matter what is the initial space, and under the assumptions

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in [17] this limit space is doubling. Hence the question of how suddenly appears the doubling property of the limit seems natural.

In this note we prove that, in a precise sense, the elements of Hutchinson orbits become more and more doubling as the step of the iteration grows (Theorem 3.2), and the doubling property of the limit, in this sense, is gradual (Proposition 3.1). We use this result to prove the uniform upper doubling property of the orbits in Theorem 4.1.

For the sake of notational simplicity we have assumed the same contractivity coefficient for each one of the contractive similitudes involved. As can easily be verified, the results in this paper still hold in the case of different coefficients.

In the first section we introduce notation and definitions, and some basic results. In Section 3 we find a gradual improvement for the doubling property of the orbit as the iteration step increases in Theorem 3.2, the proof of which is based on the construction of discrete approximations to the attractor. We consider the orbits starting from a mass point space, defined by a finite family of contractive similitudes, and we prove the uniform normality, and hence the uniform doubling property, for the whole orbit (Lemmas 3.3 and 3.4). We also state some basic properties of iterated function systems in Lemma 3.5. Finally, in Section 4, we use Theorem 3.2 in order to show that the approximating sequence of spaces is a uniform family of upper doubling spaces, in the sense of Hytönen.

## 2. NOTATION AND BASIC RESULTS

Let us start by describing our general framework. Throughout this note,  $(X, d)$  is a given compact metric space. Without loss of generality we shall assume that  $\text{diam}(X) = 1$ , where  $\text{diam}(X) = \sup\{d(x, y) : x, y \in X\}$ . We define an open ball centered at  $x$  of radius  $r$  to be the set  $B(x, r) = \{y \in X : d(x, y) < r\}$ .

Let  $\mathcal{K} = \{K \subseteq X : K \neq \emptyset, K \text{ compact}\}$ . With  $[A]_\varepsilon$  we shall denote the  $\varepsilon$ -enlargement of the set  $A \subseteq X$ ; i.e.  $[A]_\varepsilon = \bigcup_{x \in A} B(x, \varepsilon) = \{y \in X : d(y, A) < \varepsilon\}$ . Here  $d(x, A) = \inf\{d(x, y) : y \in A\}$ . Given  $A$  and  $B$  two sets in  $\mathcal{K}$  the Hausdorff distance from  $A$  to  $B$  is given by

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subseteq [B]_\varepsilon \text{ and } B \subseteq [A]_\varepsilon\}.$$

Let us now introduce the Kantorovich-Hutchinson distance on the set of all Borel regular probability measures on the metric space  $(X, d)$ . We denote

$$\mathcal{P} = \{\mu : \mu \text{ is a non-negative Borel measure on } X \text{ and } \mu(X) = 1\},$$

and  $\text{Lip}_1$  as the space of all Lipschitz continuous functions defined on  $X$  with Lipschitz constant less than or equal to one, i.e.  $f \in \text{Lip}_1$  if and only if  $|f(x) - f(y)| \leq d(x, y)$  for every  $x, y \in X$ .

Since  $(X, d)$  is compact,  $d_K(\mu, \nu) = \sup\{| \int f d\mu - \int f d\nu | : f \in \text{Lip}_1\}$  gives a distance on  $\mathcal{P}$  such that the  $d_K$ -convergence of a sequence is equivalent to its weak star convergence to the same limit (see [7] for the Euclidean case and [3] for more general settings).

In [3] is considered the following metric on  $\mathcal{K} \times \mathcal{P}$

$$\delta((Y_1, \mu_1), (Y_2, \mu_2)) = d_H(Y_1, Y_2) + d_K(\mu_1, \mu_2),$$

with  $(Y_i, \mu_i) \in \mathcal{K} \times \mathcal{P}$ ,  $i = 1, 2$ . So that  $(\mathcal{K} \times \mathcal{P}, \delta)$  becomes a complete metric space. It also is proved that the set

$$\mathcal{E} = \{(Y, \mu) \in \mathcal{K} \times \mathcal{P} : \text{supp}(\mu) \subseteq Y\}$$

is a closed subset of  $\mathcal{K} \times \mathcal{P}$ , so that  $(\mathcal{E}, \delta)$  is a complete metric space.

Throughout this paper, a finite set  $\Phi = \{\phi_i : X \rightarrow X, i = 1, 2, \dots, M\}$  of contractive similitudes with the same contractivity coefficient is given. Precisely, each  $\phi_i$  satisfies

$$d(\phi_i(x), \phi_i(y)) = \frac{1}{a}d(x, y)$$

for every  $x, y \in X$  and some  $a > 1$ . The constant  $\frac{1}{a}$  is called the *contractivity coefficient*.

We consider the transformation  $T$  induced by  $\Phi$ , defined on  $(Y, \mu) \in \mathcal{E}$  for  $T(Y, \mu) = (T_1(Y), T_2(\mu)) = (Y', \mu')$  where

$$Y' = \bigcup_{i=1}^M \phi_i(Y),$$

and

$$\mu'(B) = \frac{1}{M} \sum_{i=1}^M \mu(\phi_i^{-1}(B)),$$

for every Borel subset  $B$  of  $X$ . It is easy to see that  $T : \mathcal{E} \rightarrow \mathcal{E}$  is a  $\delta$ -contraction.

From the (Banach) fixed point theorem we have that any  $\delta$ -contractive mapping  $T : \mathcal{E} \rightarrow \mathcal{E}$  has a unique fixed point. Moreover, the fixed point can be achieved as the limit for  $n \rightarrow \infty$  of the  $n$ -th iteration  $T^n$  of  $T$  starting at any initial point  $(Y_0, \mu_0) \in \mathcal{E}$ .

Let us write  $(Y_\infty, \mu_\infty)$  to denote the unique limit point of  $T^n(Y_0, \mu_0)$ , which depends only on  $T$  but not on the starting space  $(Y_0, \mu_0)$ . The fractal set  $Y_\infty$  is called the *attractor* of the system  $\Phi$ , and is the only compact set in  $X$  satisfying

$$Y_\infty = \bigcup_{i=1}^M \phi_i(Y_\infty).$$

On the other hand  $\mu_\infty$  is called the *invariant measure* and is the only probability Borel measure supported in  $Y_\infty$  such that

$$\mu_\infty(B) = \frac{1}{M} \sum_{i=1}^M \mu_\infty(\phi_i^{-1}(B)),$$

for every Borel set  $B$  (see [7], [12]).

We shall say that the system  $\Phi = \{\phi_1, \dots, \phi_M\}$  satisfies the *open set condition* (OSC) if there exists a non-empty open set  $U \subset X$  such that

$$\bigcup_{i=1}^M \phi_i(U) \subseteq U,$$

and  $\phi_i(U) \cap \phi_j(U) = \emptyset$  if  $i \neq j$ . We shall say that  $U$  is an *open set for the OSC* for  $\Phi$  (see for example [12], [7] and [16]).

The basic examples of IFS are the systems generating the most classical and best-known fractal sets, such as the ternary Cantor set, the Sierpinski gasket and carpet, and the von Koch snowflake. For example, in the case of the ternary Cantor  $C$  set the system  $\Phi$  is defined on  $X = [0, 1]$  equipped with the Euclidean distance

$d(x, y) = |x - y|$ , and consists of  $M = 2$  contractive similitudes with contractivity coefficient equal to  $1/3$  ( $a = 3$ ). More precisely  $\phi_1(x) = x/3$  and  $\phi_2(x) = x/3 + 2/3$ . Also  $U = (0, 1)$  is an open set for the OSC for  $\Phi$ . If we take  $Y_0 = [0, 1]$  and  $\mu_0 =$  Lebesgue measure on  $Y_0$ , we have that  $T_1(Y_0) = [0, 1/3] \cup [2/3, 1]$ ,  $T_1^2(Y_0) = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$ , and in general,  $T_1^n(Y_0)$  is the union of  $2^n$  disjoint intervals of the  $n$ -th step in the usual construction of the Cantor set. Denoting this union by  $C_n$ , we have that  $T_2^n(\mu_0)$  coincides with the uniform measure on  $C_n$  normalized to a probability measure. But if we now take  $Y_0 = \{0\}$  and  $\mu_0$  to be the Dirac delta concentrated at 0, then  $T_1^n(Y_0)$  is the collection  $L_n$  of all the left points of each interval in  $C_n$ , and  $T_2^n(\mu_0)$  is the counting measure on  $L_n$  divided by  $2^n$ . As we already mentioned, the fixed point depends only on  $T$  but not on the starting space  $(Y_0, \mu_0)$ . In this case the limit  $(Y_\infty, \mu_\infty)$  is the Cantor set  $C$  with the  $s$ -dimensional Hausdorff measure on  $C$ , where  $s = \log 2 / \log 3$  (see [7], [12]).

Then we consider some subclasses of  $\mathcal{E}$  introduced in [2].

**Definition 2.1.** Given  $(Y, \mu) \in \mathcal{E}$ , we say that  $(Y, \mu)$  is a *space of homogeneous type*, or that  $\mu$  is a *doubling measure* on  $Y$ , if there exists a constant  $A \geq 1$  such that the inequalities

$$(2.1) \quad 0 < \mu(B(y, 2r)) \leq A\mu(B(y, r))$$

hold for every  $y \in Y$  and every  $r > 0$ . When  $(Y, \mu) \in \mathcal{E}$  satisfies (2.1) we shall write  $(Y, \mu) \in \mathcal{D}_A$  to keep record of the quantitative parameter of the doubling property. Set  $\mathcal{D} = \bigcup_{A \geq 1} \mathcal{D}_A$ .

We make some remarks concerning the classes  $\mathcal{D}_A$  defined above. First let us observe that  $\mathcal{D}_1$  is not empty, since every set consisting of a single point equipped with any metric and with the counting measure, belongs to  $\mathcal{D}_1$ . Notice also that if  $A_1 \leq A_2$  then  $\mathcal{D}_{A_1} \subseteq \mathcal{D}_{A_2}$ . Finally let us point out that if  $(Y, \mu) \in \mathcal{D}_A$  then  $\text{supp}(\mu) = Y$ . In fact, since  $(Y, \mu) \in \mathcal{E}$  we have  $\text{supp}(\mu) \subseteq Y$ . On the other hand, for  $y \in Y \setminus \text{supp}(\mu)$  there exists an open set  $G$  containing  $y$  with  $\mu(G) = 0$ . So that for some ball  $B$  in  $Y$  we should have  $\mu(B) = 0$ , which is impossible.

It is proved in [17] that, under the open set condition for the system  $\Phi$ , the limit set (attractor) equipped with the invariant measure and the usual Euclidean distance, is a (normal or Ahlfors regular) space of homogeneous type. With our notation  $(Y_\infty, \mu_\infty) \in \mathcal{D}$ . However, the examples in [2] show that it may happen that the only points in the orbit satisfying the doubling property are  $(Y_0, \mu_0)$  and  $(Y_\infty, \mu_\infty)$ , but no other  $(Y_n, \mu_n) := T^n(Y_0, \mu_0)$  is a space of homogeneous type. These examples seem to suggest that, if we take a sequence  $\{\varepsilon_n\}$  whose elements tend to zero when  $n \rightarrow \infty$ , then there exists a constant  $A \geq 1$  such that (2.1) holds for every  $n$ , every  $y \in Y_n$  and every  $r \geq \varepsilon_n$ . This leads us to define the  $\varepsilon$ -doubling condition for a measure and consequently another subclass of  $\mathcal{E}$ .

**Definition 2.2.** For  $\varepsilon > 0$ , we shall denote by  $\mathcal{D}_A^\varepsilon$  the class of all those couples  $(Y, \mu) \in \mathcal{E}$  for which (2.1) holds for every  $y \in Y$  and every  $r \geq \varepsilon$ . Set  $\mathcal{D}^\varepsilon = \bigcup_{A \geq 1} \mathcal{D}_A^\varepsilon$ .

We shall now give a representative example of the above definition.

**Example 2.3.** Take  $Y = [0, 1]$  and  $\mu(E) = \frac{1}{10} \text{card}\{E \cap Z\}$ , where the set  $Z$  is defined by  $Z = \{\frac{j}{10}, j = 0, 1, \dots, 9\}$ . It is easy to see that  $(Y, \mu) \notin \mathcal{D}$  since

$\mu(B(\frac{1}{20}, \frac{1}{20})) = 0$  but  $\mu(B(\frac{1}{20}, \frac{1}{10})) = \frac{1}{5}$ . Nevertheless (2.1) becomes true for every  $y \in Y$ , taking  $r \geq 1/10$ .

### 3. UNIFORM GRADUAL DOUBLING ORBITS STARTING AT ANY POINT OF $\mathcal{E}$

When the elements of the approximating sequence become more and more doubling in a sense that will be precise, then the doubling property of the limit  $(Y_\infty, \mu_\infty)$  does not appear suddenly but naturally. This result is stated in the next proposition.

**Proposition 3.1.** *Let  $((Y_n, \mu_n))_{n \geq 1}$  be a sequence in  $\mathcal{E}$  such that  $(Y_n, \mu_n) \in \mathcal{D}_A^{\varepsilon_n}$  for some sequence  $\varepsilon_n \rightarrow 0$  when  $n \rightarrow \infty$ . If  $(Y_n, \mu_n) \xrightarrow{\delta} (Y, \mu)$  then  $(Y, \mu) \in \mathcal{D}_{A^4}$ .*

*Proof.* Take  $y \in Y$  and  $r > 0$ . Let  $\psi$  be the continuous function defined on  $\mathbb{R}_0^+$  by  $\psi \equiv 1$  on  $[0, 1]$ ,  $\psi \equiv 0$  on  $[2, \infty)$ , and by assuming it to be linear on  $[1, 2]$ . For  $t > 0$  we denote  $\psi_{y,t}(x) = \psi(\frac{d(y,x)}{t})$  for  $x \in X$ . Since  $Y_n \xrightarrow{d_H} Y$ , we can choose  $y_n \in Y_n$  such that  $d(y_n, y) \rightarrow 0$  when  $n \rightarrow \infty$ . Then, since there exists  $n_0$  such that  $y_n \in B(y, r/16)$  and  $\varepsilon_n < 5r/16$  for every  $n \geq n_0$ , we have that

$$\begin{aligned} \mu(B(y, 2r)) &\leq \int \psi_{y,2r}(x) d\mu(x) \\ &= \lim_{n \rightarrow \infty} \int \psi_{y,2r}(x) d\mu_n(x) \\ &\leq \liminf_{n \rightarrow \infty} \mu_n(B(y, 4r)) \\ &\leq \liminf_{n \rightarrow \infty} \mu_n(B(y_n, 5r)) \\ &\leq \liminf_{n \rightarrow \infty} A^4 \mu_n\left(B\left(y_n, \frac{5r}{16}\right)\right) \\ &\leq A^4 \liminf_{n \rightarrow \infty} \mu_n\left(B\left(y, \frac{r}{2}\right)\right) \\ &\leq A^4 \lim_{n \rightarrow \infty} \int \psi_{y,r/2}(x) d\mu_n(x) \\ &= A^4 \int \psi_{y,r/2}(x) d\mu(x) \\ &\leq A^4 \mu(B(y, r)). \end{aligned}$$

□

Let us state the additional hypothesis that we are assuming for the results in the remainder of this section. We shall assume that  $(X, d)$  has furthermore *finite metric (or Assouad) dimension*. This means that there exists a constant  $N \in \mathbb{N}$ , called a constant for the Assouad dimension of  $X$ , such that for every  $x \in X$ , every  $r > 0$  and every  $r$ -disperse subset  $E$  of  $X$ , we have that  $\text{card}(E \cap B(x, 2r)) \leq N$ . A set  $E$  is said to be  $r$ -disperse if  $d(x, y) \geq r$  for every  $x, y \in E$ ,  $x \neq y$ . If  $(X, d)$  has finite metric dimension, then every  $r$ -disperse subset of  $X$  has at most  $N^m$  points in each ball of radius  $2^m r$ , with  $m$  a positive integer (see [6] and [4]).

The following theorem is the main result of our paper. This theorem proves that the elements of Hutchinson orbits generated by a transformation induced by IFS become uniformly more and more doubling in the following sense: the doubling

property is satisfied with the same doubling constant except, perhaps for radii which decrease to zero as the step of the iteration grows.

**Theorem 3.2.** *Let  $\Phi = \{\phi_1, \dots, \phi_M\}$  be a family of contractive similitudes on  $X$  with the same contractivity coefficient  $1/a$  and satisfying the OSC. Let  $U$  be an open set for the OSC of  $\Phi$ . Let  $(Y_0, \mu_0) \in \mathcal{E}$  such that  $Y_0 \subseteq U$ , and for each non-negative integer  $n$ , set  $(Y_n, \mu_n) = T^n(Y_0, \mu_0)$ . Then there exists a constant  $A \geq 1$  such that  $(Y_n, \mu_n) \in \mathcal{D}_A^{5a^{-n}}$  for each  $n$ .*

Notice that the above theorem and Proposition 3.1 with  $\varepsilon_n = 5a^{-n}$  show that, even when no point of the orbit is a space of homogeneous type, the doubling property of the limit space  $(Y_\infty, \mu_\infty)$  does not appear suddenly.

In order to prove Theorem 3.2, we shall use the following three lemmas. In the Lemmas 3.3 and 3.4 we assume the hypothesis  $(Y_0, \mu_0) \in \mathcal{E}$  with  $Y_0 \subseteq U$  and we use the definitions given below.

Let us fix  $u \in Y_0$ , and for each non-negative integer  $n$  set

$$T^n(\{u\}, \lambda_u) = (\Delta_n, \nu_n),$$

where  $\lambda_u$  is the unit mass at  $u$ . So that  $\Delta_n$  has  $M^n$  elements and for every  $x \in \Delta_n$  we have  $\nu_n(\{x\}) = M^{-n}$ . In other words,  $\nu_n$  is the measure on  $X$  counting the points of  $\Delta_n$ , normalized to a probability measure.

Notice that from the OSC each  $Y_n$  can be written as a disjoint union of  $M^n$  Borel pieces  $Y_n^i$ , where  $i \in \{1, 2, \dots, M\}^n$ . Also  $\Delta_n \subseteq Y_n$  and  $\text{card}(\Delta_n \cap Y_n^i) = 1$  for every  $i \in \{1, 2, \dots, M\}^n$ .

**Lemma 3.3.** *There exists a constant  $C \geq 1$ , depending on  $\rho = \text{dist}(Y_0, \partial U) > 0$ , such that the inequalities*

$$C^{-1}r^s \leq \nu_n(B(x, r)) \leq Cr^s,$$

hold for every  $r \geq \rho a^{-n}$ , every  $x \in \Delta_n$  and every  $n \in \mathbb{N}$ , where  $s = \log_a M$ .

The above lemma states that each  $(\Delta_n, \nu_n)$  is Ahlfors  $s$ -regular space for every  $r \geq \rho a^{-n}$  with constant which not depends on  $n$ .

**Lemma 3.4.** *The elements of the sequence  $((\Delta_n, \nu_n))_{n \geq 1}$  are uniformly spaces of homogeneous type. In other words, there exists a constant  $\tilde{A} \geq 1$  such that  $((\Delta_n, \nu_n))_{n \geq 1} \subseteq \mathcal{D}_{\tilde{A}}$ .*

Finally we shall state some basic results about IFS. Given  $\mathbf{i} = (i_1, i_2, \dots, i_k) \in \{1, 2, \dots, M\}^k$  we denote with  $\phi_{\mathbf{i}}$  the composition  $\phi_{i_k} \circ \phi_{i_{k-1}} \circ \dots \circ \phi_{i_2} \circ \phi_{i_1}$ . Also, if  $i_0 \in \{1, 2, \dots, M\}$  we write  $\mathbf{i}' = (i_0, \mathbf{i})$  to denote the  $(k+1)$ -tuple  $(i_0, i_1, i_2, \dots, i_k)$ .

**Lemma 3.5.** *With  $U$  an open set for the OSC for  $\Phi$ , we have*

- (a) if  $\mathbf{i}, \mathbf{j} \in \{1, 2, \dots, M\}^k$  and  $\mathbf{i} \neq \mathbf{j}$ , then  $\phi_{\mathbf{i}}(U) \cap \phi_{\mathbf{j}}(U) = \emptyset$ ;
- (b) if  $\mathbf{i} = (i, \mathbf{i}')$  with  $\mathbf{i}' \in \{1, 2, \dots, M\}^k$  and  $i \in \{1, 2, \dots, M\}$ , then  $\phi_{\mathbf{i}}(U) \subseteq \phi_{\mathbf{i}'}$ (U);
- (c) if  $\mathbf{i}'$  and  $\mathbf{j}'$  are two different elements in  $\{1, 2, \dots, M\}^k$  and  $\mathbf{i} = (i, \mathbf{i}')$  where  $i \in \{1, 2, \dots, M\}$ , then  $\phi_{\mathbf{i}}(U) \cap \phi_{\mathbf{j}'}$ (U) =  $\emptyset$ ;
- (d) for any fixed  $u \in U$  and each positive integer  $n$ , if we define

$$\Delta_n = \{\phi_{\mathbf{j}}(u) : \mathbf{j} \in \{1, 2, \dots, M\}^n\},$$

then we have that

$$\text{card}(\phi_{\mathbf{i}}(U) \cap \Delta_n) = M^{n-k}$$

for every  $k \leq n$  and every  $\ell \in \{1, 2, \dots, M\}^k$ .

The proof of Lemma 3.5 is contained in [2]. We shall postpone the proofs of the Lemmas 3.3 and 3.4 and we give the proof of the theorem.

*Proof of Theorem 3.2.* Let  $(Y_0, \mu_0) \in \mathcal{E}$  such that  $Y_0 \subseteq U$ . Notice that

$$\mu_n(Y_n^j) = \mu_n(\phi_j(Y_0)) = M^{-n}$$

for every  $n \in \mathbb{N}$  and every  $j \in \{1, 2, \dots, M\}^n$ . In fact, for a fixed  $j \in \{1, 2, \dots, M\}^n$  we have that

$$\begin{aligned} \mu_n(\phi_j(Y_0)) &= M^{-n} \sum_{i \in \{1, 2, \dots, M\}^n} \mu_0(\phi_i^{-1}(\phi_j(Y_0))) \\ &= M^{-n} \mu_0(Y_0) + M^{-n} \sum_{\substack{i \in \{1, 2, \dots, M\}^n \\ i \neq j}} \mu_0(\phi_i^{-1}(\phi_j(Y_0))). \end{aligned}$$

Since  $\mu_0(Y_0) = 1$  and  $\phi_i^{-1}(\phi_j(Y_0)) = \emptyset$  for every choice of  $i \neq j$  (see Lemma 3.5 (a)), we have the claim.

Fix  $n \in \mathbb{N}$ ,  $y \in Y_n$  and  $r \geq 5a^{-n}$ . There exists one and only one  $i \in \{1, \dots, M\}^n$  such that  $y \in Y_n^i$ . Let us write  $x_n^i$  to denote the unique point in  $\Delta_n \cap Y_n^i$ . Then  $d(y, x_n^i) \leq a^{-n}$ . For  $t > 2a^{-n}$  denote

$$B_k = B(x_n^i, t + (k-2)a^{-n}),$$

$k = 0, 1, 3, 4$ . Notice that

$$B_1 \subseteq B(y, t) \subseteq B_3,$$

and then

$$\mu_n(B_1) \leq \mu_n(B(y, t)) \leq \mu_n(B_3).$$

We claim that the comparison of the measure  $\mu_n$  with the counting measure  $\nu_n$  on  $\Delta_n$  is the following

$$(3.1) \quad \mu_n(B_1) \geq \nu_n(B_0) \quad \text{and} \quad \mu_n(B_3) \leq \nu_n(B_4).$$

If the claim holds, then

$$\nu_n(B_0) \leq \mu_n(B(y, t)) \leq \nu_n(B_4)$$

for every  $y \in Y_n^i$  and  $t > 2a^{-n}$ . Let  $\tilde{A} \geq 1$  be a constant such that  $\{(\Delta_n, \nu_n) : n \in \mathbb{N}\} \subseteq \mathcal{D}_{\tilde{A}}$  (see Lemma 3.4). Then

$$\begin{aligned} \mu_n(B(y, 2r)) &\leq \nu_n(B(x_n^i, 2r + 2a^{-n})) \\ &\leq \tilde{A}^2 \nu_n(B(x_n^i, (r + a^{-n})/2)) \\ &\leq \tilde{A}^2 \mu_n(B(y, (r + 5a^{-n})/2)) \\ &\leq \tilde{A}^2 \mu_n(B(y, r)), \end{aligned}$$

and the result holds with  $A = \tilde{A}^2$ . Then it only remains to prove the inequalities contained in (3.1). To show the first one we define the set

$$\mathcal{J} = \{j \in \{1, \dots, M\}^n : Y_n^j \subseteq B_1\}.$$

Let us prove that if  $x_n^j \in B_0 \cap \Delta_n$  then  $j \in \mathcal{J}$ . Since  $x_n^j \in B_0$  we have that  $d(x_n^j, x_n^i) < t - 2a^{-n}$ . To see that  $Y_n^j \subseteq B_1$  fix  $z \in Y_n^j$ . Since  $\text{diam}(Y_n^j) = a^{-n}$  we have that  $d(z, x_n^j) \leq a^{-n}$ . Then

$$d(z, x_n^i) \leq d(z, x_n^j) + d(x_n^j, x_n^i)$$



$$\begin{aligned} &< a^{-n} + t - 2a^{-n} \\ &= t - a^{-n}, \end{aligned}$$

and hence  $Y_n^j \subseteq B_1$ . So that

$$\begin{aligned} \mu_n(B_1) &\geq \sum_{j \in \mathcal{J}} \mu_n(Y_n^j) \\ &= \sum_{j \in \mathcal{J}} M^{-n} \\ &= \sum_{j \in \mathcal{J}} \nu_n(\{x_n^j\}) \\ &\geq \nu_n(B_0). \end{aligned}$$

To prove the second inequality let us now define the set

$$\mathcal{Q} = \{\mathbf{q} \in \{1, \dots, M\}^n : Y_n^{\mathbf{q}} \cap B_3 \neq \emptyset\}.$$

Observe that if  $\mathbf{q} \in \mathcal{Q}$  then  $Y_n^{\mathbf{q}} \subseteq B_4$ . In fact, if  $\mathbf{q} \in \mathcal{Q}$  there exists  $z_n^{\mathbf{q}} \in Y_n^{\mathbf{q}} \cap B(x_n^i, t + a^{-n})$ . Then for every  $z \in Y_n^{\mathbf{q}}$  we have

$$\begin{aligned} d(z, x_n^i) &\leq d(z, z_n^{\mathbf{q}}) + d(z_n^{\mathbf{q}}, x_n^i) \\ &< a^{-n} + t + a^{-n} \\ &= t + 2a^{-n}, \end{aligned}$$

and then  $z \in B_4$ . Hence

$$\begin{aligned} \mu_n(B_3) &\leq \sum_{\mathbf{q} \in \mathcal{Q}} \mu_n(Y_n^{\mathbf{q}}) \\ &= \sum_{\mathbf{q} \in \mathcal{Q}} \nu_n(\{x_n^{\mathbf{q}}\}) \\ &\leq \nu_n(B_4), \end{aligned}$$

as desired.  $\square$

*Proof of Lemma 3.3.* Fix  $n \in \mathbb{N}$ ,  $x \in \Delta_n = \{\phi_i(u) : i \in \{1, 2, \dots, M\}^n\}$  and  $r \geq \rho a^{-n}$ , where  $\rho = \text{dist}(Y_0, \partial U) > 0$ . Let us start with the following two remarks. The first one is that if  $\ell \in \{1, 2, \dots, M\}^k$  and  $k \leq n$ , then from Lemma 3.5 (d) we have

$$\nu_n(\phi_\ell(U)) = M^{-n} \text{card}(\phi_\ell(U) \cap \Delta_n) = M^{-k} = a^{-ks},$$

with  $s = \log_a M$ .

The second remark is that the OSC implies that  $\Delta_n$  is a  $\rho a^{-n}$ -disperse set. In fact, take  $\mathbf{j}, \mathbf{i} \in \{1, \dots, M\}^n$  with  $\mathbf{j} \neq \mathbf{i}$ , and set  $x_{n,\mathbf{j}} = \phi_{\mathbf{j}}(u)$  and  $x_{n,\mathbf{i}} = \phi_{\mathbf{i}}(u)$ . Since  $U$  is an open set, we have that  $B(u, \rho) \subseteq U$ . Then

$$B(x_{n,\mathbf{j}}, \rho a^{-n}) = \phi_{\mathbf{j}}(B(u, \rho)) \subseteq \phi_{\mathbf{j}}(U),$$

$$B(x_{n,\mathbf{i}}, \rho a^{-n}) = \phi_{\mathbf{i}}(B(u, \rho)) \subseteq \phi_{\mathbf{i}}(U),$$

and since  $\phi_{\mathbf{j}}(U)$  and  $\phi_{\mathbf{i}}(U)$  are disjoint, we have  $B(x_{n,\mathbf{j}}, \rho a^{-n}) \cap B(x_{n,\mathbf{i}}, \rho a^{-n}) = \emptyset$ .

This implies that  $d(x_{n,\mathbf{j}}, x_{n,\mathbf{i}}) \geq \rho a^{-n}$ .

Assume first that  $r > a^{-n}$ . Set  $k$  to denote the only integer less than or equal to  $n$  for which  $a^{-k} < r \leq a^{-k+1}$ . For the lower bound, with  $x = \phi_{\mathbf{i}}(u)$ ,  $\mathbf{i} = (i_1, i_2, \dots, i_n)$  and  $\mathbf{i}' = (i_{n-k+1}, i_{n-k+2}, \dots, i_n)$  we have that

$$\phi_{\mathbf{i}'}(U) \cap \Delta_n \subseteq B(x, r) \cap \Delta_n.$$

In fact, if  $y \in \phi_{i'}(U) \cap \Delta_n$  then  $y = \phi_\ell(u)$ , where

$$\ell = (\ell_1, \ell_2, \dots, \ell_{n-k}, i_{n-k+1}, i_{n-k+2}, \dots, i_n),$$

for some  $\ell_1, \ell_2, \dots, \ell_{n-k} \in \{1, 2, \dots, M\}$ . Then

$$d(x, y) \leq a^{-k} < r.$$

Hence

$$\begin{aligned} \nu_n(B(x, r)) &\geq \nu_n(\phi_{i'}(U)) \\ &= a^{-ks} \\ &\geq a^{-s}r^s. \end{aligned}$$

For the upper bound, we define

$$\mathcal{J} = \{\mathbf{j} \in \{1, 2, \dots, M\}^k : B(x, r) \cap \phi_{\mathbf{j}}(U) \neq \emptyset\}.$$

Since  $\{\phi_{\mathbf{j}}(U), \mathbf{j} \in \{1, 2, \dots, M\}^k\}$  is a covering of  $\Delta_n$  we have that

$$B(x, r) \cap \Delta_n \subseteq \bigcup_{\mathbf{j} \in \mathcal{J}} \phi_{\mathbf{j}}(U).$$

From the first remark at the beginning of the proof,

$$\begin{aligned} \nu_n(B(x, r)) &\leq \sum_{\mathbf{j} \in \mathcal{J}} \nu_n(\phi_{\mathbf{j}}(U)) \\ &= \text{card}(\mathcal{J})a^{-ks} \\ &\leq \text{card}(\mathcal{J})r^s. \end{aligned}$$

We only have to show that  $\text{card}(\mathcal{J})$  is bounded by a constant which does not depend on  $x$  and  $r$ . In order to prove it, let us identify each  $\mathbf{j} \in \mathcal{J}$  with the point  $\phi_{\mathbf{j}}(u) \in \phi_{\mathbf{j}}(U)$ , and let us define the set  $\mathcal{A} = \{\phi_{\mathbf{j}}(u) : \mathbf{j} \in \mathcal{J}\}$ . Since  $\phi_{\mathbf{j}}(U)$  are pairwise disjoint for  $\mathbf{j}$  ranging on the set of indices with fixed length, we have that  $\text{card}(\mathcal{J}) = \text{card}(\mathcal{A})$ . Notice that  $\mathcal{A} \subseteq B(x, 2r)$ . In fact, if  $\mathbf{j} \in \mathcal{J}$  then there exists  $y \in B(x, r) \cap \phi_{\mathbf{j}}(U)$ , and

$$d(\phi_{\mathbf{j}}(u), x) \leq d(\phi_{\mathbf{j}}(u), y) + d(y, x) < a^{-k} + r \leq 2r.$$

Since, being a subset of  $\Delta_k$ , the set  $\mathcal{A}$  is  $\rho a^{-k}$ -disperse, we have that

$$\text{card}(\mathcal{A}) = \text{card}(B(x, 2r) \cap \mathcal{A}) \leq \text{card}(B(x, 2a^{-k+1}) \cap \mathcal{A}) \leq N^\ell,$$

where  $\ell$  is a positive integer such that  $2^\ell \geq 2a/\rho$  and  $N$  is a constant for the finite Assouad dimension of  $X$ .

Let us finally check the case  $\rho a^{-n} \leq r \leq a^{-n}$ . Notice first that

$$\nu_n(B(x, r)) \geq \nu_n(B(x, \rho a^{-n})) \geq M^{-n} = a^{-ns} \geq r^s,$$

and on the other hand,

$$\begin{aligned} \nu_n(B(x, r)) &\leq \nu_n(B(x, a^{-n})) \\ &= M^{-n} \text{card}(\Delta_n \cap B(x, a^{-n})) \\ &\leq N^{\ell-1} a^{-ns} \\ &\leq N^{\ell-1} \rho^{-s} r^s, \end{aligned}$$

with  $\ell$  and  $N$  as before.

Hence the result holds with  $C = N^\ell \rho^{-s}$ .  $\square$

*Proof of Lemma 3.4.* Fix  $n \in \mathbb{N}$ ,  $x \in \Delta_n$  and  $r > 0$ . If  $2r < \rho a^{-n}$ , since  $\Delta_n$  is  $\rho a^{-n}$ -disperse (see proof of Lemma 3.3) we have that  $B(x, 2r) \cap \Delta_n = B(x, r) \cap \Delta_n = \{x\}$  and the result holds taking  $\tilde{A} = 1$ . Otherwise, if  $2r \geq \rho a^{-n}$ , we shall consider the cases  $r \geq \rho a^{-n}$  and  $r < \rho a^{-n}$  and in both cases we shall use Lemma 3.3. In the first case we obtain

$$\nu_n(B(x, 2r)) \leq C2^s r^s \leq C2^{2s} \nu_n(B(x, r)).$$

On the other hand, when  $r < \rho a^{-n} \leq 2r$ , since  $a^s = M$  we have

$$\begin{aligned} \nu_n(B(x, 2r)) &\leq C2^s r^s \\ &< C2^s \rho^s a^{-ns} \\ &= C2^s \rho^s M^{-n} \\ &= C2^s \rho^s \nu_n(B(x, r)). \end{aligned}$$

Hence the lemma holds with  $\tilde{A} = C2^s \rho^s$ .  $\square$

As we already mentioned, for the sake of notational simplicity we have assumed the same contractivity coefficient  $1/a$  for each contractive similitude  $\phi_i$ , for  $i = 1, \dots, M$ . Nevertheless, the results in this paper still hold in the case of different coefficient  $1/a_i$ , with  $a_i > 1$  for every  $i$ . In this case, Lemma 3.3 holds with  $a_{\max}$  instead of  $a$ , where  $a_{\max} := \max_{1 \leq i \leq M} a_i$ , proof of Lemma 3.4 follows the same lines, and Theorem 3.2 holds with  $a_{\min}$  instead of  $a$ , where  $a_{\min} := \min_{1 \leq i \leq M} a_i$ .

#### 4. UNIFORM UPPER DOUBLING PROPERTY OF THE ORBITS

A new class of metric measure spaces is introduced in [8], which generalizes the spaces of homogeneous type as well as power-bounded measures on  $\mathbb{R}^n$ , that have been in the centre of the recent developments of non-doubling harmonic analysis theory.

Given a metric space  $(X, d)$  and  $Y \subseteq X$ , a Borel measure  $\mu$  on  $Y$  is called *upper doubling* if there exist a function  $\lambda : Y \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  and a constant  $C_\lambda$  such that

$$\lambda(y, r) \leq \lambda(y, s),$$

$$\lambda(y, 2r) \leq C_\lambda \lambda(y, r),$$

$$\mu(B(y, r)) \leq \lambda(y, r),$$

for every  $y \in Y$ ,  $r > 0$  and  $s \geq r$ . The function  $\lambda$  is called *dominating function*, and we say that  $(Y, \mu)$  is an *upper doubling space*.

Then, a doubling measure is a special case of upper doubling measure, with  $\lambda(x, r) = \mu(B(x, r))$ . On the other hand, non-doubling harmonic analysis has recently been developed in the spaces  $(\mathbb{R}^n, \mu)$  with  $\mu(B(x, r)) \leq Cr^t$ , for some  $t \in (0, n]$ , which are upper doubling spaces with the dominating function  $\lambda(x, r) = Cr^t$ .

The interest of upper doubling spaces has been growing during the last few years because these spaces provide an adequate framework for an abstract extension of results of harmonic and real analysis (see for example [8], [9], [10], [11] and [13]).

In this section we shall apply Theorem 3.2 in order to show that the orbits are uniformly upper doubling spaces. We shall keep the assumption of finite metric dimension of  $(X, d)$  and the hypothesis of Theorem 3.2, and as before we shall use  $(Y_n, \mu_n)$  to denote the space  $T^n(Y_0, \mu_0)$ .

For a fixed  $n$ , we define a dominating function  $\lambda_n : Y_n \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  in the following way:

$$\lambda_n(y, r) = \begin{cases} \mu_n(B(y, r)), & \text{if } r \geq 5a^{-n}; \\ \mu_n(B(y, 5a^{-n})), & \text{if } r < 5a^{-n}. \end{cases}$$

The following result states that the orbits  $(Y_n, \mu_n)$  are uniformly upper doubling spaces. Here the “uniformity” refers to the doubling constant  $C_{\lambda_n}$  for the dominating function  $\lambda_n$ .

**Theorem 4.1.** *The sequence  $((Y_n, \mu_n))_{n \geq 1}$  is a uniform family of upper doubling spaces, in the sense that the doubling constant of all dominating functions is the constant  $A$  of Theorem 3.2.*

*Proof.* Let us fix a natural number  $n$ ,  $y \in Y_n$  and  $r > 0$ . It is clear that  $\mu_n(B(y, r)) \leq \lambda_n(y, r)$  and that  $\lambda_n(y, r) \leq \lambda_n(y, s)$  if  $s \geq r$ . Also, if  $r \geq 5a^{-n}$  from Theorem 3.2 we have that

$$\lambda_n(y, 2r) = \mu_n(B(y, 2r)) \leq A\mu_n(B(y, r)) = A\lambda_n(y, r).$$

On the other hand, if  $r < 5a^{-n}$ , we shall consider the cases  $2r < 5a^{-n}$  and  $2r \geq 5a^{-n}$ . In the first case we have  $\lambda_n(y, 2r) = \lambda_n(y, r)$ , and in the second one we have

$$\lambda_n(y, 2r) \leq \lambda_n(y, 10a^{-n}) \leq A\lambda_n(y, 5a^{-n}) = A\lambda_n(y, r).$$

So that we can take  $C_{\lambda_n} = A$  for every  $n$ .  $\square$

*Remark 4.2.* Since from Proposition 3.1 the limit measure has the doubling property with constant  $A^4$ , then  $\mu$  is trivially upper doubling with dominating function  $\lambda(y, r) = \mu(B(y, r))$  for  $y \in Y$  and for all  $r > 0$ . On the other hand, since  $A$  is greater than 1 we can conclude, using above theorem, that all spaces of the orbit and its limit space are upper doubling spaces, where each corresponding dominating function has doubling constant  $A^4$ .

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