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By

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NONLOCAL SCHRÖDINGER EQUATIONS IN METRIC MEASURE SPACES

MARCELO ACTIS, HUGO AIMAR, BRUNO BONGIOANNI, AND IVANA GÓMEZ

Abstract. In this note we consider the pointwise convergence to the initial data for the solutions of some nonlocal dyadic Schrödinger equations on spaces of homogeneous type. We prove the a.e. convergence when the initial data belongs to a dyadic version of an L^2 based Besov space. In particular we give a Haar wavelet characterization of these dyadic Besov spaces.

1. Introduction

In quantum mechanics the position of a free particle in the *space* is described by the probability density function $|\varphi|^2 = \varphi\overline{\varphi}$ with φ a solution of the *Schrödinger equation*. In the classical model, the *space* is the euclidean and the *Schrödinger equation* is the associated to the Laplace operator, i.e. $i\frac{\partial\varphi}{\partial t} = \triangle\varphi$. Hence the probability of finding the particle inside the Borel set E of the euclidean space at time t is given by $\int_E |\varphi(x,t)|^2\, dx$.

The pointwise convergence to the initial data for the classical Schrödinger equation in euclidean settings is a hard problem. It is well known that some regularity in the initial data is needed [6, 9, 11, 8, 14, 17, 15].

Nonlocal operators instead of the Laplacian in this basic model have been considered previously in the euclidean space (see for example [12] and references in [16]). The nonlocal fractional derivatives as substitutes of the Laplacian become natural objects when the space itself lacks any differentiable structure and only an analysis of order less than one can be carried out.

To introduce our problem we shall start by a simple but representative situation. Set S to denote the Sierpinski triangle in $\mathbb{R}^2$ equipped with the normalized Hausdorff measure $\mu = \mathcal{H}_s$ of order $s = \frac{\log 3}{\log 2}$. We shall consider that the particle is confined to stay in S and that the probability of finding it inside the Borel subset E of S is given by $\int_E |\varphi|^2\, d\mu$.

For each positive integer j, the set S can be written as the union of 3^j translations of the contraction of S by a factor 3^{-j}, with respect to any one of the three vertices of the convex hull of S. Except for sets of μ measure zero this covering of S is disjoint. Each one of these pieces is denoted by T_k^j, $k = 1, \dots, 3^j$. We shall also write T_1^0 to denote S. Set $\mathcal{D}^j$ to denote the family $\{T_k^j : k = 1, \dots, 3^j\}$ and $\mathscr{D} = \cup_{j\geq 0}\mathscr{D}^j$. Notice that $\mu(T_k^j) = 3^{-j}$.

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For each j and each $k = 1, \dots, 3^j$ we have that T_k^j contains and is covered by exactly three pieces $T_{k_1}^{j+1}$, $T_{k_2}^{j+1}$, $T_{k_3}^{j+1}$ of the generation $j+1$. In order to abbreviate the notation, given $T \in \mathscr{D}$ we write $T(1)$, $T(2)$ and $T(3)$ to denote these pieces of T. The three dimensional space of all functions defined in T which are constant on $T(1)$, $T(2)$ and $T(3)$ has as a basis $\{\mathcal{X}_T, \mathcal{X}_{T(1)}, \mathcal{X}_{T(2)}\}$. By orthonormalization of this basis with the inner product of $L^2(d\mu)$ keeping $3^{j/2}\mathcal{X}_T$, $T \in \mathscr{D}^j$, as the first element in the new basis, we get two other functions h_T^1 and h_T^2 such that $\{3^{j/2}\mathcal{X}_T, h_T^1, h_T^2\}$ is an orthonormal basis for that three dimensional space. Notice that the orthogonality of h_T^i with $\mathcal{X}_T$ implies that $\int h_T^i d\mu = 0$, $i = 1, 2$. To fix ideas, from the self similarity of our setting, we can take h_T^1 and h_T^2 as the corresponding scalings and translations of those associated to $T = T_1^0$. For example, if

$$h_{T_1^0}^1(x) = \sqrt{\tfrac{3}{2}}\left(\mathcal{X}_{T(1)} - \mathcal{X}_{T(2)}\right),$$
$$h_{T_1^0}^2(x) = \tfrac{1}{\sqrt{2}}\left(\mathcal{X}_{T(1)} + \mathcal{X}_{T(2)} - 2\mathcal{X}_{T(3)}\right),$$

then

$$h_{T_k^j}^1(x) = 3^{\frac{j}{2}}\sqrt{\tfrac{3}{2}}\left(\mathcal{X}_{T_k^j(1)} - \mathcal{X}_{T_k^j(2)}\right)$$

and

$$h_{T_k^j}^2(x) = 3^{\frac{j}{2}}\tfrac{1}{\sqrt{2}}\left(\mathcal{X}_{T_k^j(1)} + \mathcal{X}_{T_k^j(2)} - 2\mathcal{X}_{T_k^j(3)}\right).$$

The system $\mathscr{H}$ of all these functions h is an orthonormal basis for the subspace of $L^2(T, d\mu)$ of those functions with vanishing mean. Or, equivalently, the system $\mathscr{H} \cup \{1\}$ is an orthonormal basis for $L^2(T, d\mu)$.

For a given h in $\mathscr{H}$ we say that h is *based on* $T \in \mathscr{D}$ if $h = h_T^i$ for $i = 1$ or $i = 2$. We shall use $T(h)$ to denote the triangle in which h is based. For each $T \in \mathscr{D}$ there are exactly two wavelets $h \in \mathscr{H}$ based on T.

A space time function $w : S \times \mathbb{R} \to \mathbb{C}$ of the type

$$w(x,t) = \sum_{m=-M}^{M} \sum_{h \in \mathscr{F}} c_{m,h} e^{itm} h(x)$$

with $\mathscr{F}$ a finite part of $\mathscr{H}$, can be seen as a wave on the Sierpinski triangle. For fixed $x \in S$ we have a classical trigonometric wave in the time variable. For fixed time we have a linear combination of wavelets in the Sierpinski triangle S. In our model, the dynamics of the wave function representing the state of a quantum system in the Sierpinski triangle S, is governed by a nonlocal differential operator which we proceed to introduce.

Since the diameters of the pieces T_k^j tend to zero when $j \to \infty$, given two different points x and y in S then, sooner or later, there must be two different pieces T_x containing x and T_y containing y. Hence the number $\delta(x,y) = \min\{\mu(T) : T \in \mathscr{D} \text{ such that } x, y \in T\}$ is positive and bounded above by 1. The function δ is a distance on S provided we agree at defining $\delta(x,x) = 0$.

Notice that for a given $x \in S$ and $0 < r \leq 1$ the δ-ball centered at x with radius r, $B_\delta(x,r) = \{y \in S : \delta(x,y) < r\}$ is the largest $T \in \mathscr{D}$ containing x for which $\mu(T) < r$. Hence, for $0 < r \leq 1$ we have that $\frac{r}{3} \leq \mu(B_\delta(x,r)) \leq r$. Let us also observe that $\delta(x,y) \geq |x-y|^s$, with $s = \frac{\log 3}{\log 2}$. Of course the converse inequality is not true for general points x and y in S.

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Take now a real function f in $L^2(S,d\mu)$ and denote with $\langle f,h\rangle$ the inner product in $L^2(S,d\mu)$ of f and $h\in\mathscr{H}$. In the sense of $L^2(S,d\mu)$ we have

$$f=\sum_{h\in\mathscr{H}}\langle f,h\rangle\,h,$$

whenever $\int_S f d\mu=0$.

Set $j(h)$ to denote the integer j such that $T(h)\in\mathscr{D}^j$. From the results in [1] applied to our particular setting S and $0<\beta<1$, we have that each $h\in\mathscr{H}$ is an eigenfunction with eigenvalue $\mu(T(h))^{-\beta}=3^{j(h)\beta}$ of the operator

$$g\to D^\beta g(x)=\int_S\frac{g(x)-g(y)}{\delta(x,y)^{1+\beta}}d\mu(y)$$

defined at least for g of Lipschitz class with respect to δ, as certainly is the case of the Haar functions. In other words $D^\beta g=\sum_{h\in\mathscr{H}}3^{j(h)\beta}\,\langle g,h\rangle\,h$.

The function

$$u(x,t)=\sum_{h\in\mathscr{H}}e^{-it3^{j(h)\beta}}\,\langle f,h\rangle\,h(x),$$

defined for $x\in S$ and $t>0$ is well defined as an $L^2(S,d\mu)$ function for each $t>0$.

Observe that, at least formally, $u(x,0)=f(x)$ and

$$\begin{aligned}i\frac{\partial u}{\partial t}&=\sum_{h\in\mathscr{H}}3^{j(h)\beta}e^{-it3^{j(h)\beta}}\,\langle f,h\rangle\,h(x)\\&=\sum_{h\in\mathscr{H}}e^{-it3^{j(h)\beta}}\,\langle f,h\rangle\,D^\beta h(x)=D^\beta u(x,t)\end{aligned}$$

In other words, the wave u solves the Schrödinger type problem

$$\begin{cases}i\dfrac{\partial u}{\partial t}=D^\beta u\\u(x,0)=f(x).\end{cases}\tag{1.1}$$

In this papaer we turn our attention to more general spaces than S, the spaces of homogeneous type, where the Haar Fourier analysis and dyadic fractional derivatives can be defined (see [1]).

The paper is organized in four sections. The second one deals with the general case associated to dyadic partitions of a space of homogeneous type, and we state the main results, which are contained in Theorems 3 and 4. The third section is devoted to introduce a characterization in terms of Haar coefficients of some classes of dyadic Besov spaces. In the last section we solve (1.1) in the abstract setting and we prove the pointwise convergence results to initial data of Besov type.

2. The general setting

We shall be brief in our introduction of the basic setting. For a more detailed approach see [1]. Let (X,d,μ) be a space of homogeneous type (see [13]). Let $\mathscr{D}$ be a dyadic family in X as constructed by M. Christ in [7]. Let $\mathscr{H}$ be a Haar system for $L^2(X,\mu)$ associated to $\mathscr{D}$ as built in [2] (see also [5], [3]). Following the basic notation introduced in Section 1, we shall use $\mathscr{H}$ itself s the index set for the analysis and synthesis of signals defined on X. Precisely, by $Q(h)$ we denote the member of $\mathscr{D}$ on which h *is based*. With $j(h)$ we denote the integer scale j for which $Q(h)\in\mathscr{D}^j$.

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The system $\mathscr{H}$ is an orthonormal basis for L^2_0, where L^2_0 coincides with $L^2(X.\mu)$ if $\mu(X) = +\infty$ and $L^2_0 = \{f \in L^2 : \int_X f d\mu = 0\}$ if $\mu(X) < \infty$. For a given $Q \in \mathscr{D}$ the number of wavelets h based on Q is $\#\vartheta(Q) - 1$, where $\vartheta(Q)$ is the offspring of Q and $\#\vartheta(Q)$ is its cardinal. The homogeneity property of the space together with the metric control of the dyadic sets guarantee a uniform upper bound for $\#\vartheta(Q)$. On the other hand $\#\vartheta(Q) \geq 1$ for every $Q \in \mathscr{D}$.

Let $(X, d, \mu, \mathscr{D}, \mathscr{H})$ be given as before. For the sake of simplicity we shall assume along this paper that X itself is a quadrant for $\mathscr{D}$. We say that X itself is a quadrant if given two cubes Q and Q' we have either that Q is contained in Q' or that Q' is contained in Q. A distance in X associated to $\mathscr{D}$ can be defined by $\delta(x, y) = \min\{\mu(Q) : Q \in \mathscr{D} \text{ such that } x, y \in Q\}$ when $x \neq y$ and $\delta(x, x) = 0$. The next lemma, borrowed to [1], reflects the one dimensional character of X equipped with δ and μ.

Lemma 1 (Lemma 3.1 in [1]). *Let $0 < \varepsilon < 1$, and let Q be a given dyadic cube in X. Then, for $x \in Q$, we have*

$$\int_{X \setminus Q} \frac{d\mu(y)}{\delta(x, y)^{1+\varepsilon}} \simeq \mu(Q)^{-\varepsilon}.$$

Furthermore the integral of $\delta^{-1}(x, \cdot)$ diverges on each dyadic cube containing x and, when the measure of X is not finite, on the complement of each dyadic cube.

For a complex value function f Lipschitz continuous with respect to δ define

$$D^\beta f(x) = \int_X \frac{f(x) - f(y)}{\delta(x, y)^{1+\beta}} d\mu(y),$$

One of the key results relating the operator D^β with the Haar system is provided by the following spectral theorem contained in [1].

Theorem 2 (Theorem 3.1 in [1]). *Let $0 < \beta < 1$. For each $h \in \mathscr{H}$ we have*

$$D^\beta h(x) = m_h \mu(Q(h))^{-\beta} h(x), \tag{2.1}$$

where m_h is a constant that may depends of $Q(h)$ but there exist two finite and positive constants M_1 and M_2 such that

$$M_1 < m_h < M_2, \quad \text{for all } h \in \mathscr{H}. \tag{2.2}$$

Set $B^\lambda_2(X, \delta, \mu)$ to denote the space of those functions $f \in L^2(X, \mu)$ satisfying

$$\iint_{X \times X} \frac{|f(x) - f(y)|^2}{\delta^{1+2\lambda}(x, y)} d\mu(x) d\mu(y) < \infty.$$

The projection operator defined on L^2 onto V_0 the subspace of functions which are constant on each cube $Q \in \mathscr{D}^0$ is denoted by P_0.We are now in position to state the main results of this paper.

Theorem 3. *Let $0 < \beta < \lambda < 1$ and $u_0 \in B^\lambda_2(X, \delta, \mu)$ with $P_0 u_0 = 0$ be given. Then, the function u defined on $\mathbb{R}^+$ by*

$$u(t) = -\sum_{h \in \mathscr{H}} e^{-it m_h \mu(Q(h))^{-\beta}} \langle u_0, h \rangle h, \tag{2.3}$$

(3.a) belongs to $B^\lambda_2(X, \delta, \mu)$ for every $t > 0$;

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(3.b) solves the problem

$$\begin{cases} i\dfrac{du}{dt} = D^{\beta}u & t > 0, \\ u(0) = u_0 & \text{on } X. \end{cases} \tag{2.4}$$

Precisely, $\frac{du}{dt}$ is the Fréchet derivative of $u(t)$ as a function of $t \in (0,\infty)$ with values in $B_2^{\lambda-\beta}(X,\delta,\mu)$ and $\lim_{t\to 0^+} u(t) = u_0$ in $B_2^{\lambda}(X,\delta,\mu)$.

Theorem 4. *Let $0 < \beta < \lambda < 1$ and $u_0 \in B_2^{\lambda}(X,\delta,\mu)$ with $P_0 u_0 = 0$ be given. Then,*

(4.a) there exists $Z \subset X$ with $\mu(Z) = 0$ such that the series (2.3) defining $u(t)$ converges pointwise for every $t \in [0,1)$ outside Z;

(4.b) $u(t) \to u_0$ pointwise almost everywhere on X with respect to μ when $t \to 0$.

3. Characterization of the Besov space $B_2^{\sigma}(X,\delta,\mu)$ in terms of Haar coefficients

The aim of this section is to characterize $B_2^{\sigma}(X,\delta,\mu)$ in terms of the sequence $\{\langle f,h\rangle : h \in \mathscr{H}\}$ for $0 < \sigma < 1$.

Theorem 5. *let $0 < \sigma < 1$ be given. The space $B_2^{\sigma}(X,\delta,\mu)$ coincides with the subspace of $L^2(X,\mu)$ of those functions f for which*

$$\sum_{h\in\mathscr{H}} \frac{|\langle f,h\rangle|^2}{\mu(Q(h))^{2\sigma}} < \infty.$$

Moreover,

$$\|f\|^2_{B_2^{\sigma}(X,\delta,\mu)} \simeq \|f\|^2_{L^2(X,\mu)} + \sum_{h\in\mathscr{H}} \frac{|\langle f,h\rangle|^2}{\mu(Q(h))^{2\sigma}}.$$

Let us state some important lemmas that will be useful in for the proof of the above theorem.

Lemma 6. *Let $0 < \sigma < 1$ and $h, \widetilde{h} \in \mathscr{H}$ be given. Then*

$$\nu(h,\widetilde{h}) := \iint_{X\times X} \frac{[h(x)-h(y)][\widetilde{h}(x)-\widetilde{h}(y)]}{\delta(x,y)^{1+2\sigma}} d\mu(y)d\mu(x) = 0,$$

if $h \neq \widetilde{h}$, and

$$\nu(h,h) = \iint_{X\times X} \frac{[h(x)-h(y)]^2}{\delta(x,y)^{1+2\sigma}} d\mu(y)d\mu(x) \simeq \mu(Q(h))^{-2\sigma},$$

where the equivalence constants depend only on the geometric constants of the space.

Proof. For the first part of the proof, i.e. $\nu(h,\widetilde{h}) = 0$, when $h \neq \widetilde{h}$, we shall divide our analysis in three cases according to the relative positions of $Q(h) := Q$ and $Q(\widetilde{h}) := \widetilde{Q}$, (i) $Q = \widetilde{Q}$, (ii) $Q \cap \widetilde{Q} = \emptyset$ and (iii) $Q \subsetneqq \widetilde{Q}$.

Let us start by (i). Set $\Pi_{h\widetilde{h}}(x,y) := [h(x)-h(y)][\widetilde{h}(x)-\widetilde{h}(y)]$. Notice that for x and y in $X \setminus Q$ we have $\Pi_{h\widetilde{h}}(x,y) = 0$. On the other hand, for $x \in Q$ and $y \in X \setminus Q$ we have $\Pi_{h\widetilde{h}}(x,y) = h(x)\widetilde{h}(x)$. Hence

$$\iint_{X\times X} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) = \iint_{Q\times(X\setminus Q)} \frac{h(x)\widetilde{h}(x)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)$$

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$$
\begin{aligned}
&+ \iint_{(X\setminus Q)\times Q} \frac{h(y)\widetilde{h}(y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&+ \iint_{Q\times Q} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&= \iint_{Q\times Q} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y),
\end{aligned}
$$

since for the first term $\delta(x,y)$ is constant as a function of $x \in Q$ for y fixed in $X\setminus Q$, for the second $\delta(x,y)$ is constant as a function of $y \in Q$ for x fixed in $X \setminus Q$ and h and $\widetilde{h}$ are orthogonal. Let us prove that $\iint_{Q\times Q} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) = 0$. Since h and $\widetilde{h}$ are constant on $Q' \in \vartheta(Q)$ we have that $\Pi_{h\widetilde{h}}(x,y) = 0$ for $(x,y) \in Q' \times Q'$, hence

$$
\begin{aligned}
&\iint_{Q\times Q} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&\quad= \sum_{Q'\in\vartheta(Q)} \sum_{Q''\in\vartheta(Q)} \int_{Q''}\int_{Q'} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&\quad= \sum_{Q'\in\vartheta(Q)} \int_{Q'}\int_{Q'} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&\qquad\qquad + \sum_{Q'\in\vartheta(Q)} \sum_{\substack{Q''\in\vartheta(Q)\\ Q'\neq Q''}} \int_{Q''}\int_{Q'} \frac{\Pi_{h\widetilde{h}}(x,y)}{\mu(Q)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&\quad= \mu(Q)^{-1-2\sigma} \sum_{Q''\in\vartheta(Q)} \int_{Q''} \left(\sum_{Q'\in\vartheta(Q)} \int_{Q'} \Pi_{h\widetilde{h}}(x,y) d\mu(x) \right) d\mu(y) \\
&\quad= \mu(Q)^{-1-2\sigma} \int_Q\int_Q [h(x)\widetilde{h}(x) - h(y)\widetilde{h}(x) \\
&\qquad\qquad - h(x)\widetilde{h}(y) + h(y)\widetilde{h}(y)] d\mu(x)d\mu(y) \qquad (3.1) \\
&\quad= 0.
\end{aligned}
$$

Let us now consider the case (ii), that is $Q \cap \widetilde{Q} = \emptyset$. In this case $\Pi_{h\widetilde{h}}(x,y)$ is supported in $(\widetilde{Q}\times Q)\cup(Q\times\widetilde{Q})$. Moreover on $\widetilde{Q}\times Q$ we have $\Pi_{h\widetilde{h}}(x,y) = -h(y)\widetilde{h}(x)$ and on $Q\times\widetilde{Q}$, $\Pi_{h\widetilde{h}}(x,y) = -h(x)\widetilde{h}(y)$. Since, on the other hand $\delta(x,y) = \delta(Q,\widetilde{Q})$ which is a positive constant on the support of $\Pi_{h\widetilde{h}}$, we get

$$
\begin{aligned}
&\iint_{X\times X} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&= -\frac{1}{\delta(Q,\widetilde{Q})^{1+2\sigma}} \left\{ \iint_{\widetilde{Q}\times Q} [h(y)\widetilde{h}(x)] d\mu(x)d\mu(y) + \iint_{Q\times\widetilde{Q}} [h(x)\widetilde{h}(y)] d\mu(x)d\mu(y) \right\} \\
&= 0.
\end{aligned}
$$

Consider now the case (iii). Since $Q \subsetneq \widetilde{Q}$, then $\widetilde{h}$ is constant on Q, hence

$$
\iint_{X\times X} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) = \iint_{(X\setminus Q)\times Q} + \iint_{Q\times Q} + \iint_{Q\times(X\setminus Q)}.
$$

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Since $\widetilde{h}$ is constant on Q, $\Pi_{h\widetilde{h}}$ is identically zero on $Q\times Q$ and the second term vanishes. For the first term notice that it can be written as

$$\begin{aligned}&\int_{X\setminus Q}\left(\int_Q \frac{(-h(y))(\widetilde{h}(x)-\widetilde{h}(y_Q))}{\delta(Q,x)^{1+2\sigma}}d\mu(y)\right)d\mu(x)\\ &\qquad=-\int_{X\setminus Q}\frac{\widetilde{h}(x)-\widetilde{h}(y_Q)}{\delta(Q,x)^{1+2\sigma}}d\mu(x)\left(\int_Q h(y)d\mu(y)\right)\\ &\qquad=0,\end{aligned}$$

here y_Q denotes any fixed point in Q. For the third term we have similarly

$$\begin{aligned}&\iint_{Q\times(X\setminus Q)}\frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)\\ &\qquad=\int_{X\setminus Q}\left(\int_Q\frac{(h(x))(\widetilde{h}(x_Q)-\widetilde{h}(y))}{\delta(Q,y)^{1+2\sigma}}d\mu(x)\right)d\mu(y)\\ &\qquad=0.\end{aligned}$$

Finally we have to show that $\nu(h,h)\simeq\mu(Q(h))^{-2\sigma}$. Let $Q=Q(h)$. Notice first for $(x,y)\in(X\setminus Q)\times(X\setminus Q)$ we have $\Pi_{hh}(x,y)=0$. Hence

$$\begin{aligned}&\iint_{X\times X}\frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)\\ &\quad=\iint_{(X\setminus Q)\times Q}\frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)+\iint_{Q\times Q}\frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)\\ &\qquad\qquad+\iint_{Q\times(X\setminus Q)}\frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)\\ &\quad=2\int_{X\setminus Q}\left(\int_Q\frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}}d\mu(x)\right)d\mu(y)+\iint_{Q\times Q}\frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)\\ &\quad=2I+II.\end{aligned}$$

Let us first get an estimate for I. Notice that for any $x,z\in Q$ and $y\in X\setminus Q$, we have that $\delta(x,y)=\delta(z,y)$, hence $I=\int_{X\setminus Q}\delta(z,y)^{-(1+2\sigma)}d\mu(y)\int_Q|h(x)|^2\,d\mu(x)$, which is equivalent to $\mu(Q)^{-2\sigma}$ by Lemma 1. To get the desired bound for II, we observe that equation (3.1) holds for $h=\widetilde{h}$ also, then

$$\begin{aligned}&\iint_{Q\times Q}\frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)\\ &\qquad=\mu(Q)^{-1-2\sigma}\iint_{Q\times Q}[h^2(x)+h^2(y)-2h(x)h(y)]d\mu(x)d\mu(y)\\ &\qquad=2\mu(Q)^{-2\sigma}.\end{aligned}$$

□

For $Q\in\mathscr{D}$, set $\mathscr{H}_Q=\{h\in\mathscr{H}:Q(h)\subseteq Q\}$. Let $\mathcal{S}(\mathscr{H}_Q)$ denote the linear span of $\mathscr{H}_Q$. Since $\mathscr{H}_Q$ is countable we write ℓ^2 to denote the space $\ell^2(\mathscr{H}_Q)$ of all square summable complex sequences indexed on $\mathscr{H}_Q$. On the other hand, consider the weighted space $\mathcal{L}^2(Q\times Q):=L^2(Q\times Q,\frac{d\mu(x)d\mu(y)}{\delta(x,y)})$. The next lemma shows that

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for φ and ψ in $\mathcal{S}(\mathscr{H}_Q)$ the inner product of $\frac{\varphi(x)-\varphi(y)}{\delta(x,y)^\sigma}$ with $\frac{\psi(x)-\psi(y)}{\delta(x,y)^\sigma}$ in $\mathcal{L}^2(Q\times Q)$ is equivalent to the inner product of $\frac{\langle\varphi,h\rangle}{\mu(Q(h))^\sigma}$ with $\frac{\langle\psi,h\rangle}{\mu(Q(h))^\sigma}$ in $\ell^2(\mathscr{H}_Q)$.

Lemma 7. *Let $0<\sigma<1$ and $Q\in\mathscr{D}$ be given. For φ, ψ two functions in $\mathcal{S}(\mathscr{H}_Q)$ we have that*

$$\iint_{Q\times Q}\frac{[\varphi(x)-\varphi(y)][\psi(x)-\psi(y)]}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)=\sum_{h\in\mathscr{H}_Q}\langle\varphi,h\rangle\langle\psi,h\rangle\nu(h,h).$$

In particular,

$$\iint_{Q\times Q}\frac{|\varphi(x)-\varphi(y)|^2}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)\simeq\sum_{h\in\mathscr{H}_Q}\frac{|\langle\varphi,h\rangle|^2}{\mu(Q(h))^{2\sigma}}.$$

Proof. Since φ and ψ are in the linear span of $\mathscr{H}_Q$ we easily see that

$$[\varphi(x)-\varphi(y)][\psi(x)-\psi(y)]=\sum_{h\in\mathscr{H}_Q}\sum_{\widetilde{h}\in\mathscr{H}_Q}\langle\varphi,h\rangle\left\langle\psi,\widetilde{h}\right\rangle[h(x)-h(y)][\widetilde{h}(x)-\widetilde{h}(y)]$$

Dividing both members of the above equation by $\delta(x,y)^{1+2\sigma}$, integrating on the product space $Q\times Q$ and then applying Lemma 6 we get

$$\begin{aligned}
&\iint_{Q\times Q}\frac{[\varphi(x)-\varphi(y)][\psi(x)-\psi(y)]}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)\\
&\quad=\sum_{h\in\mathscr{H}_Q}\sum_{\widetilde{h}\in\mathscr{H}_Q}\langle\varphi,h\rangle\left\langle\psi,\widetilde{h}\right\rangle\iint_{Q\times Q}\frac{(h(x)-h(y))(\widetilde{h}(x)-\widetilde{h}(y))}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)\\
&\quad=\sum_{h\in\mathscr{H}_Q}\langle\varphi,h\rangle\langle\psi,h\rangle\iint_{Q\times Q}\frac{[h(x)-h(y)]^2}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)\\
&\quad=\sum_{h\in\mathscr{H}_Q}\langle\varphi,h\rangle\langle\psi,h\rangle\nu(h,h).
\end{aligned}$$

□

Lemma 8. *For $\psi\in\mathcal{S}(\mathscr{H})$ there exists $\varepsilon>0$ such that $\psi(x)-\psi(y)$ vanishes on $\Delta_\epsilon=\{(x,y)\in X\times X:\delta(x,y)<\varepsilon\}$.*

Proof. It is enough to check the result for $\psi=h\in\mathscr{H}$. But since h is constant on each child Q' of $Q(h)$ we have that $h(x)-h(y)=0$ on $\bigcup_{Q'\text{ child of }Q}Q'\times Q'$. On the other hand, $h(x)-h(y)=0$ for x and y both outside $Q(h)$. Hence $h(x)-h(y)$ vanishes on $\{(x,y):\delta(x,y)<\mu(Q(h))\}$. □

The next result shows that Lemma 7 extends to the case of φ in L^2.

Lemma 9. *Let Q be a cube in $\mathscr{D}$, $f\in L^2(Q,\mu)$ and $\psi\in\mathcal{S}(\mathscr{H}_Q)$. Then*

$$\iint_{Q\times Q}\frac{(f(x)-f(y))(\psi(x)-\psi(y))}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)=\sum_{h\in\mathscr{H}_Q}\langle f,h\rangle\langle\psi,h\rangle\nu(h,h).$$

Proof. Let $\mathscr{F}_n\nearrow\mathscr{H}_Q$ and $f_n=\sum_{h\in\mathscr{F}_n}\langle f,h\rangle h$. Since each $f_n\in\mathcal{S}(\mathscr{H}_Q)$ we have from Lemma 7 that

$$\iint_{Q\times Q}\frac{[f_n(x)-f_n(y)][\psi(x)-\psi(y)]}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)=\sum_{h\in\mathscr{H}_Q}\langle f,h\rangle\langle\psi,h\rangle\nu(h,h)$$

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taking n large enough such that $\mathscr{F}_n$ contains all the h's building ψ. On the other hand, since, from Lemma 8, the function $\frac{\psi(x)-\psi(y)}{\delta(x,y)^{1+2\sigma}} \in L^\infty(Q\times Q)$ and $f_n(x) - f_n(y) \to f(x)-f(y)$ in $L^2(Q\times Q)$, hence in $L^1(Q\times Q)$ we get the desired equality. □

The next result allow us to localize to dyadic cubes our characterization of the Besov space $B_2^\sigma(X,\delta,\mu)$ in terms of the Haar coefficients.

Lemma 10. *The set*

$$\triangle_1 = \{(x,y)\in X\times X : \delta(x,y)<1\} = \bigcup_{Q\in\mathscr{M}} Q\times Q,$$

where $\mathscr{M}$ is the family of all maximal dyadic cubes in $\mathscr{D}$ with $\mu(Q)<1$.

The next result is a statement of the intuitive fact that the regularity requirement additional to the L^2 integrability involved in the definition of $B_2^\sigma(X,\delta,\mu)$ is only relevant around the diagonal of $X\times X$.

Lemma 11. *For $0<\sigma<1$ an L^2 function f belongs to $B_2^\sigma(X,\delta,\mu)$ if and only if*

$$\mathscr{E}(f) = \iint_{\delta(x,y)<1} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) < \infty.$$

Moreover, the $B_2^\sigma(X,\delta,\mu)$ norm of f is equivalent to $\|f\|_2 + [\mathscr{E}(f)]^{\frac{1}{2}}$.

Proof. The "only if" is obvious. Since

$$\iint_{\delta(x,y)\geq 1} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \leq C\int_{x\in X} |f(x)|^2 \left(\int_{\delta(x,y)\geq 1} \frac{d\mu(y)}{\delta(x,y)^{1+2\sigma}}\right) d\mu(x).$$

From Lemma 1 we have that the first term in the above inequality is bounded by the L^2 norm of f, as desired. □

The following result is elementary but useful.

Lemma 12. *For an $L^2(X,\mu)$ function f the quantities $\|f\|_2^2 + \sum_{h\in\mathscr{H}} \frac{|\langle f,h\rangle|^2}{\mu(Q(h))^{2\sigma}}$ and $\|f\|_2^2 + \sum_{\substack{h\in\mathscr{H}\\ \mu(Q(h))<1}} \frac{|\langle f,h\rangle|^2}{\mu(Q(h))^{2\sigma}}$ are equivalent.*

The characterization of $B_2^\sigma(X,\delta,\mu)$ in terms of the Haar system is just a completation argument built on the result in Lemma 6.

Proof of Theorem 5. From Lemmas 8, 9 and 10 it is enough to prove that that quantities

$$\iint_{Q\times Q} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \tag{3.2}$$

and

$$\sum_{h\in\mathscr{H}_Q} \frac{|\langle f,h\rangle|^2}{\mu(Q(h))^{2\sigma}} \tag{3.3}$$

are equivalent for each cube Q in $\mathscr{M}$ with constants independent of Q.

Let us start by showing that the double integral in (3.2) is bounded by the sum (3.3). Let $(\mathscr{F}_n)$ be an increasing sequence of finite subfamilies of $\mathscr{H}_Q$ that covers $\mathscr{H}_Q$. In other words $\mathscr{F}_n\subseteq\mathscr{F}_{n+1}$ and $\mathscr{H}_Q = \bigcup_n \mathscr{F}_n$. Let $f_n = \sum_{h\in\mathscr{F}_n} \langle f\chi_Q, h\rangle\, h$. Since f_n converge pointwise almost everywhere for x in Q to $f(x)$, then we have

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that $|f_n(x) - f_n(y)| \to |f(x) - f(y)|$ as $n \to \infty$ for $y \in Q$ also. Hence from Fatou's lemma and Lemma 7

$$\begin{aligned}\iint_{Q\times Q} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) &= \iint_{Q\times Q} \lim_{n\to\infty} \frac{|f_n(x)-f_n(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)\\ &\le \liminf_{n\to\infty} \iint_{Q\times Q} \frac{|f_n(x)-f_n(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)\\ &\le C \liminf_{n\to\infty} \sum_{h\in\mathscr{H}_Q} \frac{|\langle f_n, h\rangle|^2}{\mu(Q(h))^{2\sigma}}\\ &\le C \sum_{h\in\mathscr{H}_Q} \frac{|\langle f, h\rangle|^2}{\mu(Q(h))^{2\sigma}}.\end{aligned}$$

To prove the converse we shall argue by duality. Take $b = \{b_h : h \in \mathscr{H}_Q\}$ a sequence of scalars indexed on $\mathscr{H}_Q$ with $b_h = 0$ except for a finite number of $h \in \mathscr{H}_Q$. Assume that $\sum_{h\in\mathscr{H}_Q} |b_h|^2 \le 1$. Set $\psi = \sum_{h\in\mathscr{H}_Q} b_h \nu_(h,h)^{-1/2} h \in \mathcal{S}(\mathscr{H}_Q)$. So that from Lemma 9,

$$\begin{aligned}&\sum_{h\in\mathscr{H}_Q} \langle f, h\rangle \nu(h,h)^{\frac12} b_h = \sum_{h\in\mathscr{H}_Q} \langle f, h\rangle \langle \psi, h\rangle \nu(h,h)\\ &= \iint_{Q\times Q} \frac{(f(x)-f(y))(\psi(x)-\psi(y))}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)\\ &\le \left(\iint_{Q\times Q} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)\right)^{\frac12} \left(\iint_{Q\times Q} \frac{|\psi(x)-\psi(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)\right)^{\frac12}.\end{aligned}$$

Notice that from Lemma 7 and Lemma 6 we have,

$$\begin{aligned}\iint_{Q\times Q} \frac{|\psi(x)-\psi(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) &\le C \sum_{h\in\mathscr{H}_Q} \frac{|\langle \psi, h\rangle|^2}{\mu(Q(h))^{2\sigma}}\\ &\le C \sum_{h\in\mathscr{H}_Q} |b_h|^2 \left[\frac{\nu(h,h)^{-\frac12}}{\mu(Q(h))^{\sigma}}\right]^2\\ &\le C.\end{aligned}$$

Hence

$$\left|\sum_{h\in\mathscr{H}_Q} \langle f, h\rangle \nu(h,h)^{\frac12} b_h\right| \le C \left(\iint_{Q\times Q} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)\right)^{\frac12},$$

so that

$$\left(\sum_{h\in\mathscr{H}_Q} |\langle f, h\rangle|^2 \nu(h,h)\right)^{\frac12} \le C \left(\iint_{Q\times Q} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)\right)^{\frac12}$$

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or

$$\left(\sum_{h\in\mathscr{H}_Q}\frac{|\langle f,h\rangle|^2}{\mu(Q(h))^{2\sigma}}\right)^{\frac{1}{2}}\leq C\left(\iint_{Q\times Q}\frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}}d\mu(x)d\mu(y)\right)^{\frac{1}{2}}$$

as desired. □

4. Proof of Theorems 3 and 4

Before start proving the main results, let us mention that from the result of the previous section the Sobolev type condition required in [1, Theorem 4.1] is the same as our current hypothesis $f\in B_2^\lambda(X,\delta,\mu)$ when $p=2$. Then for any function $f\in B_2^\sigma(X,\delta,\mu)$ we can write

$$D^\sigma f(x)=\sum_{h\in\mathscr{H}}m_h\mu(Q(h))^{-\sigma}\,\langle f,h\rangle\,h(x),$$

where the series converges in $L^2(X,\mu)$.

Proof of Theorem 3. From Theorem 5 we see that for each $t>0$, $u(t)$ defined in (2.3) belongs to $B_2^\lambda(X,\delta,\mu)$, since $u_0\in B_2^\lambda(X,\delta,\mu)$. Moreover, for $t,s\geq 0$,

$$\begin{aligned}\|u(t)-u(s)\|^2_{B_2^\lambda(X,\delta,\mu)}&=\left\|\sum_{h\in\mathscr{H}}\left(e^{itm_h\mu(Q(h))^{-\beta}}-e^{ism_h\mu(Q(h))^{-\beta}}\right)\langle u_0,h\rangle\,h\right\|^2_{B_2^\lambda(X,\delta,\mu)}\\&=\sum_{h\in\mathscr{H}}\left|e^{itm_h\mu(Q(h))^{-\beta}}-e^{ism_h\mu(Q(h))^{-\beta}}\right|^2|\langle u_0,h\rangle|^2\\&\quad+\sum_{h\in\mathscr{H}}\left|e^{itm_h\mu(Q(h))^{-\beta}}-e^{ism_h\mu(Q(h))^{-\beta}}\right|^2\frac{|\langle u_0,h\rangle|^2}{\mu(Q(h))^{2\lambda}}.\end{aligned}$$

In order to see that both series above tend to zero as $s\to t$, for each positive ε we can take a finite subfamily $\mathscr{F}$ of $\mathscr{H}$ such that $\sum_{h\in\mathscr{H}\setminus\mathscr{F}}|\langle u_0,h\rangle|^2\left(1+\mu(Q(h))^{-2\lambda}\right)<\varepsilon$. For the sums on $\mathscr{F}$ we can argue with the continuity of the complex exponential.

Let us prove that the formal derivative of $u(t)$ is actually the derivative in the sense of $B_2^{\lambda-\beta}(X,\delta,\mu)$. In fact, for $t>0$ and τ small enough,

$$\begin{aligned}&\left\|\frac{u(t+\tau)-u(t)}{\tau}-i\sum_{h\in\mathscr{H}}e^{itm_h\mu(Q(h))^{-\beta}}\mu(Q(h))^{-\beta}\,\langle u_0,h\rangle\,h\right\|^2_{B_2^{\lambda-\beta}(X,\delta,\mu)}\\&=\left\|\sum_{h\in\mathscr{H}}e^{itm_h\mu(Q(h))^{-\beta}}\left[\frac{e^{i\tau m_h\mu(Q(h))^{-\beta}}-1}{\tau}-i\mu(Q(h))^{-\beta}\right]\langle u_0,h\rangle\,h\right\|^2_{B_2^{\lambda-\beta}(X,\delta,\mu)}\\&\leq C\left\{\left\|\sum_{h\in\mathscr{H}}e^{itm_h\mu(Q(h))^{-\beta}}\left[\frac{e^{i\tau m_h\mu(Q(h))^{-\beta}}-1}{\tau}-i\mu(Q(h))^{-\beta}\right]\langle u_0,h\rangle\,h\right\|^2_{L^2}\right.\\&\qquad\left.+\sum_{h\in\mathscr{H}}\left|\frac{e^{i\tau m_h\mu(Q(h))^{-\beta}}-1}{\tau}-i\mu(Q(h))^{-\beta}\right|^2\frac{|\langle u_0,h\rangle|^2}{\mu(Q(h))^{2(\lambda-\beta)}}\right\}\end{aligned}$$

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$$\leq C \sum_{h\in\mathscr{H}} \mu(Q(h))^{2\beta} \left| \frac{e^{i\tau m_h \mu(Q(h))^{-\beta}} - 1}{\tau} - i\mu(Q(h))^{-\beta} \right|^2 \frac{|\langle u_0, h\rangle|^2}{\mu(Q(h))^{2\lambda}}.$$

Since, from Theorem 5, $\sum_{h\in\mathscr{H}} \frac{|\langle u_0,h\rangle|^2}{\mu(Q(h))^{2\lambda}} < \infty$ and

$$\mu(Q(h))^{2\beta} \left| \frac{e^{i\tau m_h \mu(Q(h))^{-\beta}} - 1}{\tau} - i\mu(Q(h))^{-\beta} \right|^2 = \left| \frac{e^{i\tau m_h \mu(Q(h))^{-\beta}} - 1}{\mu(Q(h))^{-\beta}\tau} - i \right|^2$$

tends to zero as $\tau \to 0$ for each $h \in \mathscr{H}$, arguing as before we obtain the result.

On the other hand since $u(t) \in B_2^\lambda(X,\delta,\mu)$ and since $\lambda > \beta$, $D^\beta u(t)$ is well defined and it is given by

$$\begin{aligned} D^\beta u(t) &= D^\beta \left(\sum_{h\in\mathscr{H}} e^{itm_h \mu(Q(h))^{-\beta}} \langle u_0, h\rangle h \right) \\ &= \sum_{h\in\mathscr{H}} e^{itm_h \mu(Q(h))^{-\beta}} \mu(Q(h))^{-\beta} \langle u_0, h\rangle h = -i\frac{du}{dt}. \end{aligned}$$

Hence $u(t)$ is a solution of the nonlocal equation and *(3.b)* is proved.

□

Before proving Theorem 4 we shall obtain some basic maximal estimates involved in the proofs of *(4.a)* and *(4.b)*. With M_{dy} we denote the Hardy-Littlewood dyadic maximal operator given by

$$M_{dy} f(x) = \sup \frac{1}{\mu(Q)} \int_Q |f(y)|\, d\mu(y)$$

where the supremum is taken on the family of all dyadic cubes $Q \in \mathscr{D}$ for which $x \in Q$. Calderón's dyadic maximal operator of order λ is defined by

$$M^\#_{\lambda,dy} f(x) = \sup \frac{1}{\mu(Q)^{1+\lambda}} \int_Q |f(y) - f(x)|\, d\mu(y),$$

where the supremum is taken on the family of all dyadic cubes Q of X such that $x \in Q$. The following lemma can be seen as an extension of a result by DeVore and Sharpley in [10, Corollary 11.6].

Lemma 13. *If $f \in B_2^\lambda(X,\delta,\mu)$, then $\left\| M^\#_{\lambda,dy} f \right\|_{L^2} \leq \|f\|_{B_2^\lambda}$.*

Proof. Let $Q \in \mathscr{D}$ and $x \in Q$ be given. Applying Schwarz's inequality, since $\delta(x,y) \leq \mu(Q)$ for $y \in Q$, we have

$$\begin{aligned} \frac{1}{\mu(Q)^{1+\lambda}} \int_Q |f(y) - f(x)|\, d\mu(y) &\leq \frac{1}{\mu(Q)^{1+\lambda}} \left(\int_Q |f(y) - f(x)|^2\, d\mu(y) \right)^{\frac{1}{2}} \mu(Q)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{\mu(Q)^{1+2\lambda}} \int_Q |f(y) - f(x)|^2\, d\mu(y) \right)^{\frac{1}{2}} \\ &\leq \left(\int_X \frac{|f(y) - f(x)|^2}{\delta(x,y)^{1+2\lambda}} d\mu(y) \right)^{\frac{1}{2}}. \end{aligned}$$

□

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In this section we shall need a better description of the structure of $\mathscr{H}$ in terms of scales. As we said before $\mathscr{D}^j$ denotes the dyadic sets at the scale $j \in \mathbb{Z}$. With $\mathscr{H}^j$ we denote the Haar wavelets which are based in the cubes of $\mathscr{D}^j$ for j a fixed integer. Since from our construction of the Haar system we have that

$$V_{j+1} = V_j \oplus W_j,$$

then $\mathscr{H}^j$ is an orthonormal basis for W_j. Hence, for $N > 0$, $V_{N+1} = V_0 \oplus W_0 \oplus \ldots \oplus W_N$. So that, for $P_0 f = 0$, the projection of f onto V_{N+1} is given by

$$P_{N+1} f = \sum_{j=0}^{N} \mathcal{Q}_j f,$$

where $\mathcal{Q}_j$ is the projector onto W_j, precisely

$$\mathcal{Q}_j f = \sum_{h \in \mathscr{H}^j} \langle f, h \rangle \, h.$$

As it is easy to see, since V_{N+1} is the space of those L^2 functions on X which are constant on each cube of $\mathscr{D}^{N+1}$, $|P_{N+1} f(x)| \leq M_{dy} f(x)$, pointwise.

Let us next introduce two maximal operators related to the series (2.3). For fixed $t > 0$ set

$$S_t^* f(x) = \sup_{N \in \mathbb{N}} \left| S_t^N f(x) \right|$$

with

$$S_t^N f(x) = \sum_{j=0}^{N} \sum_{h \in \mathscr{H}^j} e^{-itm_h \mu(Q(h))^{-\beta}} \, \langle f, h \rangle \, h(x).$$

Set

$$S^* f(x) = \sup_{0<t<1} S_t^* f(x).$$

Lemma 14. *Let $f \in B_2^\lambda(X, \delta, \mu)$ with $0 < \beta < \lambda < 1$ and $P_0 f = 0$. Then, the inequalities*

(14.a) $S_t^* f(x) \leq C t M_{\lambda, dy}^{\#} f(x) + 2 M_{dy} f(x)$ for $t \geq 0$ and $x \in X$;
(14.b) $S^* f(x) \leq C M_{\lambda, dy}^{\#} f(x) + 2 M_{dy} f(x)$ for $x \in X$;
(14.c) $\left\| S^* f \right\|_{L^2} \leq C \left\| f \right\|_{B_2^\lambda(X, \delta, \mu)}$,
hold for some constant C which does not depend on f.

Proof. For $f \in B_2^\lambda(X, \delta, \mu)$, $t \geq 0$ and $N \in \mathbb{N}$, we have

$$\left| S_t^N f(x) \right| \leq \left| S_t^N f(x) - S_0^N f(x) \right| + \left| S_0^N f(x) \right|. \tag{4.1}$$

Since $S_0^N f(x) = P_N f(x)$, we have $\sup_N \left| S_0^N f(x) \right| \leq M_{dy} f(x)$. Let us now estimate the first term on the right hand side of (4.1). Let $Q(j, x)$ be the only cube Q in $\mathscr{D}^j$ for which $x \in Q$. Let $\mathscr{H}(j, x)$ be the set of all the wavelets based on $Q(j, x)$. Recall that $\#(\mathscr{H}(j, x))$ is bounded by a purely geometric constant. Then

$$\begin{aligned} &\left| S_t^N f(x) - S_0^N f(x) \right| \\ &= \left| \sum_{j=0}^{N} \sum_{h \in \mathscr{H}(j,x)} \left[e^{-itm_h \mu(Q(j,x))^{-\beta}} - 1 \right] \langle f, h \rangle \, h(x) \right| \end{aligned}$$

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$$= \left| \sum_{j=0}^{N} \sum_{h\in\mathscr{H}(j,x)} \left[e^{-itm_h\mu(Q(j,x))^{-\beta}} - 1 \right] \int_{Q(j,x)} (f(y)-f(x))h(y)d\mu(y)\, h(x) \right|$$

$$= C\, t \sum_{j=0}^{N} \sum_{h\in\mathscr{H}(j,x)} \frac{\left| e^{-itm_h\mu(Q(j,x))^{-\beta}} - 1 \right|}{m_h t \mu(Q(j,x))^{-\lambda}} \frac{1}{\mu(Q(j,x))^{1+\lambda}} \int_{Q(j,x)} |f(y)-f(x)|\, d\mu(y)$$

$$\leq C\, t \sum_{j=0}^{N} \mu(Q(j,x))^{\lambda-\beta} M^{\#}_{\lambda,dy} f(x)$$

$$\leq \widetilde{C}\, t M^{\#}_{\lambda,dy} f(x)$$

and *(14.a)* is proved. Since $t<1$, *(14.b)* follows. And applying Lemma 13 we get *(14.c)*. □

Proof of Theorem 4. Since for each $Q\in\mathscr{D}$ we have that $\mathcal{X}_Q$ belongs to $Lip(X,\delta)$, in fact $|\mathcal{X}_Q(x)-\mathcal{X}_Q(y)|\leq \frac{\delta(x,y)}{\mu(Q)}$, then $\mathcal{S}(\mathscr{H})$ is a dense subspace of $L^2(X,\mu)$ such that $S_t^N g(x)$ converges as $N\to\infty$ for every $x\in X$ and every $t\geq 0$. From lemmas 13, 14, and the above remark the result of Theorem 4 follows the standard argument of pointwise convergence (see [4]). □

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