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APPROXIMATION OF SOLUTION OF FRACTIONAL DIFFUSIONS IN COMPACT METRIC MEASURE SPACES

MARCELO ACTIS AND HUGO AIMAR

ABSTRACT. In this note we prove that the solutions to diffusions associated to fractional powers of the Laplacian in compact metric measure spaces can be obtained as limits of the solutions to particular rescalings of some nonlocal diffusions with integrable kernels. The abstract approach considered here has several particular and interesting instances.

1. Introduction

The Cauchy problem for the heat equation in $\mathbb{R}^n$, i.e. $u_t = \Delta u$ in $\mathbb{R}^{n+1}_+$, with $u(x,0) = u_0$ in $\mathbb{R}^n$, admits an immediate generalization to the case of nonlocal diffusions. In this case, the Laplacian in the spatial variables is replaced by the fractional Laplacian operator of order s with $0 < s < 2$, which is given by

$$-(-\Delta)^{s/2} f(x) = c_{n,s}\, v.p. \int \frac{f(x) - f(y)}{|x-y|^{n+s}}\, dy, \tag{1.1}$$

and is a representation of the generalized *Dirichlet to Neumann operator* (see [6]). The standard linear evolution equation $u_t = -(-\Delta)^{s/2} u$ involving the fractional Laplacian have been widely studied and usually used in modeling processes like anomalous diffusion (see [15] and the references therein).

The aim of this paper is to approach the study of fractional diffusions in metric measure spaces where despite of the lack of differential structure in these contexts problems associated to nonlocal operators can be considered. As it is explicitly observed in [5], usually the solutions to nonlocal evolution equations with integrable kernels approximate solutions of some classical local evolution problems such as the heat equation (see [9]). What we do here is to extend this basic principle both to nonlocal and to non-Eeuclidean settings. For a related approach in the euclidean case see [10, 11].

Let us observe that there are several settings where the application of our main result, contained in Theorem 8, can provide good approximation of solutions. In particular we shall deal with two applications in the Section 5 of this paper:

- The unit circle of the complex plane.
- Dyadic compact metric measure space.

The above seemingly diverse settings can be unified in their approach by noticing that all of them have the same structural form in Ahlfors regular spaces. We say that a metric measure space (X, d, μ) is an Ahlfors α-regular space if there exists two positive constants, say c_1 and c_2, such that

$$c_1 r^\alpha \leq \mu(B(x,r)) \leq c_2 r^\alpha,$$

for all $x \in X$ and for all real number $r \leq \operatorname{diam}(X)$, where $B(x,r)$ denotes the d-ball centered in x of radius r and $\operatorname{diam}(X) = \sup\{d(x,y) : x, y \in X\}$ denotes the diameter of the whole space X.

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In this generalized context the fractional diffusion problem takes the form

$$\begin{cases} u_t(x,t) = -D^s u(x,t), & x \in X, t \in (0,T), \\ u(x,0) = u_0(x), & x \in X, \end{cases} \tag{1.2}$$

where D^s is the natural extension given by (1.1) of the fractional Laplacian to Ahlfors α-regular spaces, i.e.

$$D^s f(x) = \int_X \frac{f(x) - f(y)}{d(x,y)^r} .$$

Let $J : X \times X \to \mathbb{R}^+$ be a non-negative measurable function with respect to the product σ-algebra in $X \times X$ satisfying the following properties

(i) $J(x,y) = J(y,x)$, for all $x, y \in X$;

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(ii) J is integrable in each variable uniformly in the other, i.e.

$$\int_X J(x,y)d\mu(y) \leq \beta, \quad \forall x \in X.$$

(iii) $J(\cdot,y)$ is Lipschitz continuous of order $r>0$ uniformly in $y \in X$, i.e.

$$[J(\cdot,y)]_{\Lambda_r} \leq \lambda, \quad \forall y \in X.$$

Notice that in the case of X bounded the L^∞-norm is controlled by de Lipschitz seminorm, then property (iii) implies (ii).

Given $T \in \mathbb{R}^+$ fixed and $u_0 \in \Lambda_r(X,d,\mu)$ we consider the nonlocal problem

$$\begin{cases} u_t(x,t) = \int_X J(x,y)[u(y,t)-u(x,t)]d\mu(y), & x \in X, t \in [0,T], \\ u(x,0) = u_0(x), & x \in X, \end{cases} \tag{2.2}$$

We say that a function u is a *solution* of (2.2) if u belongs to

$$\mathbb{B}_{\Lambda_r} = C^1((0,T);\Lambda_r(X,d,\mu)) \cap C(([0,T];\Lambda_r(X,d,\mu))$$

and satisfies

$$u(x,t) = u_0(x) + \int_0^t \int_X J(x,y)(u(y,s)-u(x,s))\,d\mu(y)\,ds,$$

where the integral in the right hand side is formally understood as a Bochner integral. Existence and uniqueness of solutions of problem (2.2) are consequence of the Banach fixed point theorem and will be shown in Theorem 3. First, let us state two auxiliary lemmas.

Given $t_0>0$, let $\mathbb{X}_{t_0}$ be the space of continuous functions from $[0,t_0]$ to $\Lambda_r(X,d,\mu)$, i.e.

$$\mathbb{X}_{t_0} = C([0,t_0];\Lambda_r(X,d,\mu)),$$

which is a Banach space once is equipped with the norm

$$|||w|||_r = \max_{t\in[0,t_0]} \|w(\cdot,t)\|_{\Lambda_r}.$$

For any $w_0 \in \Lambda_r(X,d,\mu)$, let T_{w_0} be the operator defined on $\mathbb{X}_{t_0}$ by

$$T_{w_0}(w)(x,t) = w_0(x) + \int_0^t \int_X J(x,y)(w(y,s)-w(x,s))\,d\mu(y)\,ds. \tag{2.3}$$

Lemma 1. *The operator T_{w_0} maps $\mathbb{X}_{t_0}$ into $\mathbb{X}_{t_0}$.*

Proof. Notice that for any $t \in \mathbb{R}^+$ and any $x,z \in X$ we have that

$$\begin{aligned} |T_{w_0}(w)(x,t) - T_{w_0}(w)(z,t)| &\leq \left| \int_0^t \int_X [J(x,y)-J(z,y)]w(y,s)\,d\mu(y)\,ds \right| \\ &\quad + \left| \int_0^t \int_X [J(x,y)-J(z,y)]w(z,s)\,d\mu(y)\,ds \right| \\ &\quad + \left| \int_0^t \int_X J(x,y)[w(x,s)-w(z,s)]\,d\mu(y)\,ds \right| \\ &\leq 2t\mu(X) \sup_{y\in X}[J(\cdot,y)]_{\Lambda_r} \sup_{t\in[0,t_0]} \|w(\cdot,t)\|_\infty d(x,z)^r \\ &\quad + t\mu(X) \int_X J(x,y)d\mu(y) \sup_{t\in[0,t_0]} [w(\cdot,t)]_{\Lambda_r} d(x,z)^r \end{aligned}$$

Hence, by properties (ii) and (iii), we obtain

$$|T_{w_0}(w)(x,t) - T_{w_0}(w)(z,t)| \leq Ct\, |||w|||_r\, d(x,z)^r. \tag{2.4}$$

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Then $[T_{w_0}(w)(\cdot,t)-w_0]_{\Lambda_r} \leq Ct$, which proves the continuity at $t=0$. Analogously, if $t_1, t_2 \in \mathbb{R}^+$ such that $0 < t_1 < t_2 \leq 0$ then we obtain that

$$[T_{w_0}(w)(\cdot,t_1) - T_{w_0}(w)(\cdot,t_2)]_{\Lambda_r} \leq C(t_1 - t_2).$$

So clearly $T_{w_0}(w) \in \mathbb{X}_{t_0}$. □

Lemma 2. *Let $w, v \in \mathbb{X}_{t_0}$ then*

$$|||T_{w_0}(w) - T_{w_0}(v)|||_r \leq Ct_0 \, |||w-v|||_r \, .$$

Proof. Let $0 < t < t_0$ and $u := w-v$. Notice that $[T_{w_0}(w)-T_{w_0}(v)]_{\Lambda_r} = [T_{w_0}(u)]_{\Lambda_r}$. Since by (2.4)

$$|T_{w_0}(u)(x,t) - T_{w_0}(u)(z,t)| \leq Ct \, |||u|||_r \, d(x,z)^r,$$

then $[T_{w_0}(u)]_{\Lambda_r} \leq Ct \, |||u|||_r$. Therefore $[T_{w_0}(w) - T_{w_0}(v)]_{\Lambda_r} \leq Ct_0 \, |||w-v|||_r$ as desired. □

Theorem 3 (Existence and uniqueness)**.** *Let $u_0 \in \Lambda_r(X,d,\mu)$ and J satisfying* (i), (ii) *and* (ii). *Then there exists a unique solution $u \in \mathbb{B}_{\Lambda_r}$ of* (2.2).

Proof. Taking t_0 in Lemma 2 satisfying $Ct_0 < 1$ we obtain that T_{u_0} is a contractive operator on $\mathbb{X}_{t_0}$. Then the existence and uniqueness of a solution satisfying (2.2) follows from the Banach fixed point theorem on the interval $[0,t_0]$.

To extend the solution to $[0,T]$, we take as initial data $u(x,t_0) \in \Lambda_r(X,d,\mu)$ to obtain a solution up to $[0,2t_0]$. Iterating this process we get a solution defined on $[0,T]$. □

Finally, we present our last preliminary result. It is a comparison principle which shall be useful at proving our main result.

We say that a function $u \in \mathbb{B}_C = C^1((0,T),C(X)) \cap C([0,T],C(X))$ is a *supersolution* of (2.2) if

$$\begin{cases} u_t(x,t) \geq \displaystyle\int_X J(x,y)[u(y,t)-u(x,t)]\,d\mu(y), & x \in X, t \in (0,T), \\ u(x,0) \geq u_0(x), & x \in X. \end{cases}$$

Lemma 4 (Comparison principle)**.** *Let $u \in \mathbb{B}_C$ be a supersolution of* (2.2) *with initial datum $u_0 \in C(X)$ and such that $u_0 \geq 0$. Then $u \geq 0$.*

Proof. Suppose that u is negative somewhere. Let $v(x,t) = u(x,t) + \epsilon t$, with ϵ small enough to make v negative at some point. So, if (x_0,t_0) is the point where v reaches its minimum, then $t_0 > 0$ (since $v(x,0) = u(x,0) \geq 0$) and further

$$\begin{aligned} v_t(x_0,t_0) = u_t(x_0,t_0) + \epsilon &> \int_X J(x,y)[u(y,t_0) - u(x,t_0)]d\mu(y) \\ &= \int_X J(x,y)[v(y,t_0) - v(x,t_0)]d\mu(y) \geq 0. \end{aligned}$$

Therefore, $v_t(x_0,t_0) > 0$ which contradicts the fact that (x_0,t_0) is a point where v reaches its minimum. Then, $u \geq 0$. □

3. Approximation of D^s by rescaling kernels

In this section (X,d,μ) is an Ahlfors α-regular space, i.e. there exists two positive constants c_1 and c_2 such that

$$c_1 r^\alpha \leq \mu(B(x,r)) \leq c_2 r^\alpha, \tag{3.1}$$

for all $x \in X$ and for all $r < 2\,\mathrm{diam}(X)$. This situation, although restrictive, is natural in lots of classic geometric contexts as manifolds and even fractals coming from iterated function systems like the Cantor set or the Sierpinski gasket (see [13]).

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The first result in this section is an elementary lemma which reflects the α dimensional character of X under the assumption (3.1). For the sake of notational simplicity we shall write $A \simeq B$ when the quotient A/B is bounded above and below by positive and finite constants. In a similar way we write $A \lesssim B$ when A/B is bounded above.

Lemma 5. *Let (X, d, μ) be an Ahlfors α-regular space. Then for any $\delta > 0$ and any $r > 0$ we have that*

$$\int_{X\setminus B(x,r)} \frac{d\mu(y)}{d(x,y)^{\alpha+\delta}} \simeq r^{-\delta}$$

and

$$\int_{B(x,r)} \frac{d\mu(y)}{d(x,y)^{\alpha-\delta}} \simeq r^{\delta},$$

where the hidden constants only depend on α and δ.

Proof. In order to prove the second estimate let us rewrite $B(x,r)$ as the union of annuli of the form $A_j = B(x, 2^{-(j-1)}r)\setminus B(x, 2^{-j}r)$, with $j \in \mathbb{N}$. Hence

$$\begin{aligned}\int_{B(x,r)} \frac{d\mu(y)}{d(x,y)^{\alpha-\delta}} &= \sum_{j=1}^{\infty} \int_{A_j} \frac{d\mu(y)}{d(x,y)^{\alpha-\delta}} \\ &\leq \sum_{j=1}^{\infty} \left(2^{-j}r\right)^{-\alpha+\delta} \int_{A_j} d\mu(y) \\ &\leq \sum_{j=1}^{\infty} \left(2^{-j}r\right)^{-\alpha+\delta} \mu(B(x, 2^{-(j-1)}r)).\end{aligned}$$

Therefore, by the upper bound in (3.1) we obtain

$$\int_{B(x,r)} \frac{d\mu(y)}{d(x,y)^{\alpha-\delta}} \leq c_2 \sum_{j=1}^{\infty} \left(2^{-j}r\right)^{-\alpha+\delta} (2^{-(j-1)}r)^{\alpha} \leq c_2 \frac{2^{\alpha}}{2^{\delta}-1} r^{\delta}.$$

In an analogous way it can be proved the lower bound and also the estimate over $X\setminus B(x,r)$. □

The fractional derivative operator D^s, with $0 < s < 1$, given by

$$D^s f(x) = \int_X \frac{f(x) - f(y)}{d(x,y)^{\alpha+s}} \, d\mu(y),$$

is well defined for $f \in \Lambda_r(X, d, \mu)$, with $s < r \leq 1$. Indeed, if we call $B := B(x,1)$ then

$$\int_X \frac{f(x) - f(y)}{d(x,y)^{\alpha+s}} \, d\mu(y) = \int_B \frac{f(x) - f(y)}{d(x,y)^{\alpha+s}} \, d\mu(y) + \int_{X\setminus B} \frac{f(x) - f(y)}{d(x,y)^{\alpha+s}} \, d\mu(y). \tag{3.2}$$

Since f satisfies (2.1) then

$$\int_B \frac{f(x) - f(y)}{d(x,y)^{\alpha+s}} \, d\mu(y) \leq [f]_{\Lambda_r} \int_B \frac{d\mu(y)}{d(x,y)^{\alpha-(r-s)}}.$$

Then from Lemma 5 we obtain

$$\int_B \frac{f(x) - f(y)}{d(x,y)^{\alpha+s}} \, d\mu(y) \lesssim [f]_{\Lambda_r}. \tag{3.3}$$

To estimate the integral over $X\setminus B$ of (3.2) we use the fact that f is bounded and again the Lemma 5,

$$\int_{X\setminus B} \frac{f(x) - f(y)}{d(x,y)^{\alpha+s}} \, d\mu(y) \leq 2\|f\|_{L^\infty} \int_{X\setminus B} \frac{d\mu(y)}{d(x,y)^{\alpha+s}} \lesssim \|f\|_{L^\infty}. \tag{3.4}$$

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Therefore, from (3.2), (3.3) and (3.4) we get that

$$|D^s f(x)| \lesssim \|f\|_{\Lambda_r}.$$

To build the approximations to D^s, take $\psi : \mathbb{R}^+ \to \mathbb{R}_0^+$ defined by

$$\psi(t) = \begin{cases} 1, & \text{si } t < 1, \\ t^{-\alpha-s}, & \text{si } t \geq 1. \end{cases}$$

For each $0 < \epsilon \leq 1$ we define a kernel J_ϵ in the following way,

$$J_\epsilon(x,y) := \frac{1}{\epsilon^\alpha}\psi\left(\frac{d(x,y)}{\epsilon}\right). \tag{3.5}$$

Lemma 6. *The kernels J_ϵ defined by* (3.5) *are symmetric and positive. Moreover,*

$$\int_X J_\epsilon(x,y)\, d\mu(y) \simeq C, \tag{3.6}$$

where C is a constant independent of ϵ and for $0 < r \leq 1$

$$[J_\epsilon(\cdot, y)]_{\Lambda_r} \lesssim \epsilon^{-(\alpha+r)}. \tag{3.7}$$

Proof. The symmetry and the positivity are inherited from the distance d and the function ψ, respectively. Besides,

$$\begin{aligned}\int_X J_\epsilon(x,y)\, d\mu(y) &= \frac{1}{\epsilon^\alpha}\int_X \psi\left(\frac{d(x,y)}{\epsilon}\right) d\mu(y) \\ &= \frac{1}{\epsilon^\alpha}\left[\int_{B(x,\epsilon)} d\mu(y) + \epsilon^{\alpha+s}\int_{X\setminus B(x,\epsilon)} \frac{d\mu(y)}{d(x,y)^{\alpha+s}}\right].\end{aligned}$$

Hence, by (3.1) and Lemma 5 we obtain (3.6).

On the other hand, since $|\psi'| \leq \alpha + s$ we have

$$\begin{aligned}|J_\epsilon(x,y) - J_\epsilon(z,y)| &= \frac{1}{\epsilon^\alpha}\left|\psi\left(\frac{d(x,y)}{\epsilon}\right) - \psi\left(\frac{d(z,y)}{\epsilon}\right)\right| \\ &\leq \frac{[\psi]_{\Lambda_r}}{\epsilon^\alpha}\left|\frac{d(x,y)}{\epsilon}\right|^r \\ &\leq \frac{\alpha+s}{\epsilon^{\alpha+r}} d(x,y)^r,\end{aligned}$$

which implies (3.7), so the proof is completed. □

Finally let L_ϵ be an operator given by

$$L_\epsilon f(x) = \frac{1}{\epsilon^s}\int_X J_\epsilon(x,y)[f(y) - f(x)]\, d\mu(y).$$

The next statement shows that L_ϵ converge weakly to D^s as $\epsilon \to 0$.

Theorem 7 (Weak approximation)**.** *Let $f \in \Lambda_r(X,d,\mu)$ then*

$$\sup_{x\in X} |L_\epsilon f(x) - D^s f(x)| \lesssim [f]_{\Lambda_r}\epsilon^{r-s}.$$

Proof. Since $f \in \Lambda_r(X,d,\mu)$ then $D^s f$ and $L_\epsilon f$ are well defined. Furthermore notice that

$$\left|\frac{1}{\epsilon^s}J_\epsilon(x,y) - k_s(x,y)\right| \leq \frac{\chi_{\{d(x,y)<\epsilon\}}}{d(x,y)^{\alpha+s}}.$$

Hence we obtain that

$$\begin{aligned}|L_\epsilon f(x) - D^s f(x)| &= \left|\int_X \left[\frac{1}{\epsilon^s}J_\epsilon(x,y) - k_s(x,y)\right][f(y) - f(x)]\, d\mu(y)\right| \\ &\leq \int_{B(x,\epsilon)} \frac{|f(y) - f(x)|}{d(x,y)^{\alpha+s}}\, d\mu(y)\end{aligned}$$

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$$\leq [f]_{\Lambda_r} \int_{B(x,\epsilon)} \frac{d\mu(y)}{d(x,y)^{\alpha-(r-s)}}.$$

Thus by Lemma 5 we get that

$$|L_\epsilon f(x) - D^s f(x)| \lesssim [f]_{\Lambda_r} \epsilon^{r-s}$$

and so the result is immediate. □

4. Main result

For each $\epsilon \in (0,1)$ the kernel J_ϵ defined in (3.5) satisfies (ii) and (iii), then the problem

$$\begin{cases} u_t(x,t) = L_\epsilon u(x,t), & x \in X, t \in (0,T) \\ u(x,0) = u_0(x), & x \in X. \end{cases} \tag{4.1}$$

has a unique solution $u^\epsilon \in \mathbb{B}_{\Lambda_r}$.

The next theorem shows that, provided a solution v of problem 1.2 then the solutions u^ϵ of the problems (4.1) converge to v as $\epsilon \to 0^+$.

Theorem 8. *Let $u_0 \in \Lambda_r(X,d,\mu)$ and let $s, r \in \mathbb{R}$ be such that $0 < s < r \leq 1$. Suppose there exists a solution $v(x,t) \in \mathbb{B}_{\Lambda_r}$ of the problem*

$$\begin{cases} v_t(x,t) = -D^s v(x,t), & x \in X, t \in (0,T), \\ v(x,0) = u_0(x), & x \in X. \end{cases} \tag{4.2}$$

Then the solutions u^ϵ of the problems (4.1) satisfy

$$|||v - u^\epsilon|||_\infty := \sup_{t\in[0,T]} \sup_{x\in X} |v(x,t) - u^\epsilon(x,t)| \lesssim T\epsilon^{r-s}.$$

Proof. Let $w^\epsilon = v - u^\epsilon$. Observe that

$$\begin{cases} w^\epsilon_t(x,t) = L_\epsilon w^\epsilon(x,t) + F_\epsilon(x,t), & x \in X, t \in (0,T), \\ w^\epsilon(x,0) = 0, & x \in X, \end{cases}$$

where $F_\epsilon(x,t) = D^s v(x,t) - L_\epsilon v(x,t)$.

Define $\overline{z} = k\epsilon^{r-s}t - w^\epsilon$, where k is a arbitrary constant. Observe that

$$\overline{z}_t(x,t) = k\epsilon^{r-s} - w^\epsilon_t(x,t) = k\epsilon^{r-s} - (L_\epsilon w^\epsilon(x,t) + F_\epsilon(x,t)).$$

We already know by Theorem 7 that $|F_\epsilon(x,t)| \lesssim \epsilon^{r-s}$. Thus, choosing k large enough we have that $k\epsilon^\theta - F_\epsilon(x,t) \geq 0$. Then

$$\overline{z}_t(x,t) \geq -L_\epsilon w^\epsilon(x,t) = L_\epsilon \overline{z}(x,t).$$

Therefore $\overline{z}$ is a supersolution of the problem (4.1). Since $\overline{z}(x,0) = 0$ then by Lemma 4 it turns out that $^-$

$(x,t) = k\epsilon^{r-s}t + w^\epsilon(x,t)$ we can prove that $\underline{z}(x,t) \geq 0$ and so $w^\epsilon(x,t) \geq k\epsilon^{r-s}t$. Thereby

$$|v(x,t) - u^\epsilon(x,t)| = |w^\epsilon(x,t)| \leq k\epsilon^{r-s}t$$

which implies that

$$\sup_{t\in[0,T]} \sup_{x\in X} |v(x,t) - u^\epsilon(x,t)| \lesssim T\epsilon^{r-s}.$$

□

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The abstract approach considered here has several particular and interesting instances. One main disadvantage is that we required the a priori existence of solution of problem (4.2). However, we believe that is possible to prove the Cauchy character of the approximant sequence even when no assumption on the existence of solution for the fractional diffusion is made.

5. Applications

Let us finally provide two quite different settings in which our main result applies. The first one is the very classical case of the unit circle and the second is the abstract situation provided by dyadic metrics on spaces of homogeneous type. Let us start by the classical case. Set S^1 to denote the unit circle of the complex plane and D to denote the disc with boundary S^1. The Poisson kernel written in its series expansion in D is given by

$$P_r(\theta) = \sum_{k\in\mathbb{Z}} r^{|k|} e^{ik\theta},$$

where $0 < r < 1$ and θ parametrizes S^1 with $-\pi < \theta < \pi$. For the unit disc the direction of the outer normal to S^1 coincides with the direction of increasing of the variable r. Hence the normal derivative of the harmonic function $P_r(\theta)$ at the boundary of the disc can be computed as $\frac{\partial P_r}{\partial r}\big|_{r=1}$. This procedure gives the natural "first order" differential operator

$$\frac{\partial P_r}{\quad}$$

$$1 - 2r\cos\theta + r^2 .$$

Taking the partial derivative of $P_r(\theta)$ with respect to r in this formula and taking then the value of this derivative for $r = 1$, we can easily get a new expression for the Neumann condition for $P_r(\theta)$ on S^1. Namely

$$\frac{\partial P_r}{\partial r}(\theta)\bigg|_{r=1} = C\frac{1}{1-\cos\theta}.$$

Now, from cosine theorem, if $d(z, z')$ denotes the restriction to S^1 of the euclidean distance on $\mathbb{R}^2$ we have with $z = e^{i\theta}$

$$d^2(z, 1) = 2(1 - \cos\theta).$$

Hence the Neumann condition for the Poisson kernel of the disc can be written as

$$\frac{\partial P}{\partial \vec{n}}(z) = \frac{C}{d^2(z,1)},$$

which has the expected behavior in terms of the metric for the first order non local operator on S^1. Of course this extend naturally, since (S^1, d, length) is a bounded 1-Ahlfors regular space to any $0 < s < 1$ by the kernel $d(z,1)^{-(1+s)}$.

Our second application deals with the operator considered in [1] in the bounded case. Let us briefly introduce the setting. For a more detailed approach see [1].

Let (X, d, μ) be a compact space of homogeneous type (see [12]). Let $\mathscr{D}$ be a dyadic family in X as constructed by M. Christ in [7]. Let $\mathscr{H}$ be a Haar system for $\mathbb{L}^p(X,\mu) = \{f \in L^p(X,\mu) : \int_X f\, d\mu = 0\}$ associated to $\mathscr{D}$ as built in [4]. The system $\mathscr{H}$ is an unconditional basis for $\mathbb{L}_p(X,\mu)$, for $1 < p < \infty$ (see [4]). By $Q(h)$ we denote the dyadic cube on which h *is based*, i.e. the smallest member of $\mathscr{D}$ containing the set $\{x \in X : h(x) \neq 0\}$.

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A distance in X associated to $\mathscr{D}$ can be defined by $\delta(x,y) = \min\{\mu(Q) : Q \in \mathscr{D}$ such that $x, y \in Q\}$ when $x \neq y$ and $\delta(x,x) = 0$. The space X equipped with δ and μ turns out to be a 1-Ahlfors regular space. Then the fractional differential operator of order s, with $0 < s < 1$, given by

$$D^s f(x) = \int_X \frac{f(x) - f(y)}{\delta(x,y)^{1+s}} d\mu(y)$$

is well defined for every Lipschitz continuous function of order r with respect to δ, for $s < r \leq 1$.

In this context, the solution of problem (4.2) is given by

$$v(x,t) = \sum_{h \in \mathscr{H}} e^{-m_h \mu(Q(h))^{-s} t} \langle u_0, h \rangle h(x), \tag{5.1}$$

where m_h are bounded above and below by positive constants (see Theorem 4.2 in [1]). If the initial datum u_0 belongs to $\Lambda_r(X, \delta, \mu)$ it can easily be prove that v also belongs to $\Lambda_r(X, \delta, \mu)$ for every $t > 0$ (see Theorem 1.2 in [3]).

6. Conclusions

We have presented a new approach to approximate the solution of diffusions associated to fractional powers of the Laplacian as limits of the solutions to particular rescalings of some nonlocal diffusions with integrable kernels. The theory is valid in a general setting of metric measure spaces, which include fractals, manifolds and domains of $\mathbb{R}^n$ as particular cases. We proved error estimates in $L^\infty([0,T]; L^\infty(X,\mu))$ whenever the initial datum belongs to a Lipschitz spaces with regularity greater than the order of the fractional derivative.

We also have studied some existence theorems for nonlocal diffusions associated to integrable and Lipschitz kernels and a comparison principle.

The abstract approach considered here has several particular and interesting instances. One main disadvantage is that we required the a priori existence of solution of problem (4.2). However, we believe that is possible to prove the Cauchy character of the approximant sequence even when no assumption on the existence of solution for the fractional diffusion is made. We did not dwell on this matter in this article, but rather on the proposal of a first approximation method for fractional diffusions on Ahlfors regular spaces, and the proof of error estimates.

As a future work, we want to combine the result obtained in this work with the numerical method recently develop in [2] to approximate the solution of nonlocal diffusion problem like (4.1). Therefore we will be able to approximate numerically the solution of the fractional diffusion problem on a general setting of compact metric measure spaces.

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