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By

Bruno Bongioanni – Eleonor Harboure – Pablo Quijano

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Director de Publicaciones: Dr. Oscar Salinas
E-mail: salinas@santafe-conicet.gov.ar

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BOUNDEDNESS AND COMPACTNESS FOR COMMUTATORS OF SINGULAR INTEGRALS RELATED TO A CRITICAL RADIUS FUNCTION

B. BONGIOANNI, E. HARBOURE AND P. QUIJANO

Abstract. We work in the general framework of a family of singular integrals with kernels controlled in terms of a critical radius function ρ. This family models the harmonic analysis derived from the Schrödinger operator $L = -\Delta + V$, where the non-negative potential V satisfies an appropriate reverse Hölder condition. For their commutators, we find sufficient conditions on the symbols for boundedness and/or compactness when acting on weighted L^p spaces. In all cases, the classes of symbols and weights are larger than their classical counterparts, BMO, CMO and A_p. When these general results are applied to the Schrödinger context, we obtain boundedness and compactness for commutators of operators like $\nabla L^{-1/2}$, $\nabla^2 L^{-1}$, $V^{1/2}L^{-1/2}$, $V^{1/2}\nabla L^{-1}$, VL^{-1} and $L^{i\alpha}$. As in Uchiyama's classical paper, we give a full description of the class for compactness, CMO_ρ^∞, assuming ρ to be bounded. Finally, we provide examples showing that CMO is strictly contained in CMO_ρ^∞ for any ρ, bounded or not.

1. Introduction

Our main purpose in this work is to study the behaviour of commutators in the frame of the harmonic analysis related to the Schrödinger differential operator in $\mathbb{R}^d$ with $d > 2$ as given in [17], that is,

$$Lu = -\Delta u + V\,u\,,$$

where the potential V is a non-negative locally integrable function belonging to RH_q for some $q > d/2$. We remind that the last property means that there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B V^q\right)^{1/q} \leq C\frac{1}{|B|}\int_B V, \tag{1}$$

holds for any ball $B \subset, \mathbb{R}^d$. When the left hand side is replaced by $\sup_B V$, we say that $V \in \mathrm{RH}_\infty$.

In the classical harmonic analysis associate to the Laplacian, commutators of singular integrals with multiplication by a locally integrable function b were first considered by Calderón in an effort to construct an algebra of operators preserving L^p spaces. Let us remind that by commutator of a linear operator T with multiplication by a function $b \in L^1_{\mathrm{loc}}$, called symbol, means

$$T_b f(x) = [T, b]\, f(x) = T(bf)(x) - b(x)Tf(x)$$

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Notice that if T is bounded on L^p, commutators with L^∞ symbols are clearly bounded since both terms are. Nevertheless, from the cancellation we may expect commutators to preserve L^p for a wider class of symbols. In a famous work, Coifman-Rochberg and Weiss ([10]) prove that if the functions b are in BMO then commutators with Calderón-Zygmund singular integrals are bounded on L^p, $1 < p < \infty$. Furthermore, if commutators with all the Riesz transforms are bounded, the symbol b must belong to BMO. For applications to some problems related to elliptic diferential equations, it is desirable to know when such operators are also compact. In [19], Uchiyama provides an answer for commutators with singular integrals, characterizing the symbols for compactness on L^p as the functions that are limit in BMO of smooth and boundedly supported symbols. He also gives a description of this class, which he calls CMO, as those functions in BMO satisfying the following three conditions

(a) $\lim_{\delta\to 0} \sup\left\{\operatorname{Osc}(f, B(x,r)) : x \in \mathbb{R}^d, r \leq \delta\right\} = 0$
(b) $\lim_{\lambda\to \infty} \sup\left\{\operatorname{Osc}(f, B(x,r)) : x \in \mathbb{R}^d, r \geq \lambda\right\} = 0$
(c) $\lim_{|x|\to\infty} \operatorname{Osc}(f, B(x,r)) = 0$, for each fixed $r > 0$.

Here by $\operatorname{Osc}(f, B(x,r))$ we mean

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |f - f_B|,$$

where f_B stands for the average of f over the ball $B = B(x,r)$.

More recently, in [9], the authors have extended Uchiyama's result to $L^p(w)$, $1 < p < \infty$, for symbols also in CMO and w a Muckenhoupt weight in the A_p class.

In the Schrödinger context, results regarding boundedness of commutators on L^p spaces were given in [5] for the first order Riesz Transform and its adjoint, that is, for $\nabla L^{-1/2}$ and $L^{-1/2}\nabla$, showing that symbols may belong to a class larger than BMO. Later, in [7] weighted inequalities for such commutators have also been obtained. Related results can be found in [12], [8], [15] and [20].

As for compactness, in view of the above results for continuity, we may expect to get results for symbols in a class larger than CMO. However we have not found references to this question in the literature and it will be one of our main concerns here.

Let us start giving more precise details on the environment we will be working. As we said, we consider the Schrödinger differential operator with a potential satisfying a reverse Hölder condition given by (1) with $q > d/2$ and $d > 2$. In this setting the underlying idea is that L can be seen as a perturbation of the Laplacian operator and this fact can be expressed through a certain critical radius function introduced by Shen. Namely, we define $\rho : \mathbb{R}^d \longmapsto (0.\infty)$ as

$$\rho(x) = \sup\left\{ r : \frac{r^2}{|B(x,r))|}\int_{B(x,r)} V \ \leq 1\right\}. \tag{2}$$

He also proves that it satisfies the following key inequalities

$$c_\rho^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N_0} \ \leq\ \rho(y)\ \leq\ c_\rho\,\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{N_0}{N_0+1}}, \tag{3}$$

for some constant c_ρ independent of x and y.

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In [17], it is shown that the following estimates hold for the fundamental solution of L : For any positive N there exists a constant C_N such that

$$0 \leq \Gamma(x,y) \leq \frac{C_N}{|x-y|^{d-2}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}$$

That behaviour suggests that in the region $\{(x,y) : |x-y| < \rho(x)\}$ behaves as the fundamental solution of the Laplacian, while it has a much better decay at infinity scaled with the function ρ.

On the other hand, due to the light condition imposed to V, we may have not pointwise estimates for the kernels of, for instance, the Riesz transforms, neither for their size nor for smoothness; rather we may have just mean estimates. Also, smoothness estimates may hold with respect to one of the variables and not for the other. For that reason, even for L^p-spaces, $1 < p < \infty$, some operators may not be bounded over all the whole range of p. Such different behaviour leads to introduce appropriate spaces, weights and symbols which are defined in terms of the function ρ.

Along this line, in [6], suitable classes of weights are introduced as follows: we say that $w \in A_p^{\rho,\infty}$ if for some $\theta \geq 0$ there exists a constant C such that

$$w(B)^{1/p}[w^{-1/(p-1)}(B)]^{1/p'} \leq C\left(1+\frac{r}{\rho(x)}\right)^{\theta}|B| \tag{4}$$

for any ball B. Here and all along, p' denotes the conjugate exponent of p.

Appropriate family of symbols for commutators of Schrödinger Riesz Transforms have been introduced in [5], as those locally integrable functions b such that for some $\theta \geq 0$ there is a constant C satisfying

$$\mathrm{Osc}_\rho^\theta(b, B(x,r)) = \left(1+\frac{r}{\rho(x)}\right)^{-\theta}\frac{1}{|B|}\int_B |b-b_B| \leq C, \tag{5}$$

for any ball $B = B(x,r)$. For a fixed θ we call this class BMO_ρ^θ and BMO_ρ^∞ to the union over $\theta \geq 0$. Notice that BMO_ρ^θ, if we identify functions differing in a constant, turns to be a normed space under the norm

$$\|b\|_{\theta,\rho} = \sup_{B(x,r)} \mathrm{Osc}_\rho^\theta(b, B(x,r)) \tag{6}$$

Next we introduce the family of symbols to get $L^p(w)$ compactness, which is a suitable substitute for Uchiyama's class CMO. More precisely we say that a symbol b is in CMO_ρ^∞ if for some $\theta \geq 0$ there exists a sequence of $\mathcal{C}_0^\infty$ symbols $\{b_n\}_{n\in\mathbb{N}}$ such that

$$b = \lim_{n\to\infty} b_n,$$

where the limit is taken in the sense of the norm in BMO_ρ^θ, that is,

$$\sup_{B(x,r)} \mathrm{Osc}_\rho^\theta(b_n - b, B(x,r)) \to 0 \tag{7}$$

when $n \to \infty$.

Let us observe that if we take $\theta = 0$ in (4) as well as in (5) and (7), we recover Muckenhoupt classes and the spaces BMO and CMO respectively. So the new classes of symbols contain their classical counterparts but, as we shall see, they are in fact larger.

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More importantly, we remark that the above definitions do not depend on the potential V directly, only through the function ρ. Furthermore, all the properties of the above classes, $A_p^{\rho,\infty}$ and BMO_ρ^∞ are proved using the inequalities for ρ contained in (3) above.

So our strategy will be working with a general family of operators related to a given critical radius function ρ, that is $\rho : \mathbb{R}^d \to (0,\infty)$ satisfying (3). When this function derives from a Schrödinger differential operator whose potential satisfies the aforementioned conditions, our family contains all the operators we are interested in. In this general setting, we will prove our results for boundedness and compactness of commutators of such operators with symbols and weights as defined above, all in terms of ρ (see Theorems 1 and 2 below). Then, we apply those results to operators related to the Schrödinger setting to obtain $L^p(w)$ boundedness and compactness for commutators of operators such as all first and second order Riesz transforms with symbols in BMO_ρ^∞ and CMO_ρ^∞ respectively. We remark that regarding boundedness, many of the particular cases, for instance first order Riesz transforms, were already known (see [7]). Nevertheless, our analysis includes commutators of the second order Riesz transforms $\nabla^2 L^{-1}$ and $V^{1/2}\nabla L^{-1}$ which, to our knowledge, are new.

Now we formally introduce the family of operators we are going to deal with. As a motivation we recall that, in the Schrödinger context, Shen proved that some Riesz transforms are not always L^p bounded for all values of p. Rather, very often, they are bounded over some finite interval $(1,p_0]$ and hence their adjoints are bounded over $[p_0',\infty)$. Therefore such operators are not Calderón-Zygmund in the classical sense. In fact, their kernels have just integral estimates, showing singularity at the diagonal like in the Calderón-Zygmund case but with a better behaviour at infinity. This behaviour leads us to introduce, as in [4], the following class of operators modelling the adjoints of Schrödinger-Riesz Transforms. Again our definition will depend only on the function ρ and all the properties of such operators can be derived from inequalities (3), so we may forget about the Schrödinger context for a while.

Suppose we are given a critical radius function ρ. Then for each $1 < s < \infty$ and $0 < \delta \leq 1$ we shall say that a linear operator T is a **ρ-Schrödinger-Calderón-Zygmund operator of type** (s,δ) if

(I_s) T is bounded from $L^{s'}$ into $L^{s',\infty}$.

(II_s) T has an associated kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, in the sense that

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\,dy, \quad f \in L_c^{s'} \text{ and } x \notin \operatorname{supp} f.$$

Further, for each $N > 0$ there exists a constant C_N such that

$$\left(\frac{1}{R^d}\int_{R<|x_0-y|<2R} |K(x,y)|^s dy\right)^{1/s} \leq C_N R^{-d}\left(1+\frac{R}{\rho(x)}\right)^{-N}, \tag{8}$$

for $|x-x_0| < R/2$, and there exists C such that

$$\left(\frac{1}{R^d}\int_{R<|x_0-y|<2R} |K(x,y)-K(x_0,y)|^s dy\right)^{1/s} \leq C R^{-d}\left(\frac{r}{R}\right)^{\delta}, \tag{9}$$

for $|x-x_0| < r \leq \rho(x_0)$, $r < R/2$.

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Besides, we shall say that a linear operator T is a **ρ-Schrödinger-Calderón-Zygmund operator of type** (∞,δ) if the above conditions are satisfied for all s, $1<s<\infty$, meaning that T is of weak type (s',s') and the kernel satisfies (8) and (9) for any $1<s<\infty$.

Remark 1. By taking a logarithmic convex combination of (8) and (9) it is possible to obtain, for any $0<\delta'<\delta$ and every $N>0$,
(10)

$$\left(\frac{1}{R^d}\int_{R<|x_0-y|<2R}|K(x,y)-K(x_0,y)|^s dy\right)^{1/s} \le CR^{-d}\left(\frac{r}{R}\right)^{\delta'}\left(1+\frac{R}{\rho(x)}\right)^{-N},$$

for $|x-x_0|<r\le\rho(x_0)$, $r<R/2$. We refer to Lemma 4 in [4] for details.

Remark 2. We also point out that if T is of weak type (s',s') for all $1<s<\infty$ and its kernel satisfies the following pointwise inequalities

$$|K(x,y)|\le\frac{C_N}{|x-y|^d}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} \tag{11}$$

$$|K(x,y)-K(z,y)|\le\frac{C|x-z|^\delta}{|x-y|^{d+\delta}}, \tag{12}$$

as long as $|x-z|<|x-y|/2$, then T is a ρ-Schrödinger-Calderón-Zygmund operator of type (∞,δ), since (11) and (12) imply (8) and (9) for any finite s. In fact weak type for any s' (that certainly implies boundedness in L^p for all $1<p<\infty$) may be replaced by weak type $(1,1)$ (see [4]). Nevertheless, in all the concrete examples any of those requirements will be fulfilled.

Now we are in position to state the main results of this work.

Theorem 1. *Given a critical radius function ρ, let T be a ρ-Schrödinger-Calderón-Zygmund operator of type (s,δ) for $1<s\le\infty$ and $\delta>0$ and let b be a BMO_ρ^∞ symbol. Then the commutator $T_b=[T,b]$ is a bounded operator on $L^p(w)$ for any $p>s'$ and $w\in A_{p/s'}^{\rho,\infty}$.*

Theorem 2. *Given a critical radius function ρ, let T be a ρ-Schrödinger-Calderón-Zygmund operator of type (s,δ) for $1<s\le\infty$ and $\delta>0$ and let b be a CMO_ρ^∞ symbol. Then the commutator $T_b=[T,b]$ is a compact operator on $L^p(w)$ for any $p>s'$ and $w\in A_{p/s'}^{\rho,\infty}$.*

With the aid of these two general theorems, we will be able to obtain, in the Schrödinger context, continuity and compactness for commutators of operators like $L^{-1/2}\nabla$, $L^{-1}\nabla^2$, $L^{-\gamma}V^\gamma$, $L^{-\gamma}\nabla V^{1-2\gamma}$ and $L^{i\alpha}$. Then, since $L^p(w)$ are Banach spaces, we can derive continuity and compactness results over the dual spaces for Schrödinger Riesz Transforms themselves. The precise statements will be given in Section 4.

Let us remark that in [13], the authors consider a family of operators whose kernels have also integral conditions, but their size and smoothness conditions are tailored for classical singular integrals. In fact they obtain compactness results for commutators with symbols in CMO, as in Uchiyama's work. Even though they applied such compactness result to commutators with operators related to the Schrödinger semigroup, the stronger decay of the kernel in terms of the function

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ρ is not taken into account, so that the class of symbols remains the same. For related results see also [14].

Later, in Section 5, we examine the question of whether a description similar to classic CMO may be obtained for our new space CMO^∞_ρ. Under some mild extra assumption on ρ (we assume ρ to be a bounded function) we can prove that suitable conditions in the Uchiyama's spirit characterize functions in CMO^∞_ρ. Let us mention that when V is a positive polynomial, the associate ρ falls into our scope so, in particular, that characterization is valid for the Hermite operator $H = -\Delta + |x|^2$. We also provide examples showing that, for a general function ρ, there are symbols in CMO^∞_ρ that are not in CMO. In fact such examples are not even in BMO, proving also that BMO is properly contained in BMO^∞_ρ for any given critical function ρ. To our knowledge, last assertion is new, although examples were known for some special cases of ρ.

We finish this work with a comment on how boundedness and compactness of commutators of non-linear operators such as maximal operators or square functions could be handled by using the theory of vector valued operators.

2. Boundedness for commutators of ρ-Schrödinger-Calderón-Zygmund Operators

In this section we discuss Theorem 1. First we are going to give some definitions and previous results that we need in order to give a proof.

As usual, a weight means a non-negative, locally integrable function. As we mentioned before, given a critical radius function ρ we will consider classes of weights that properly contain the Muckenhoupt A_p classes. Following [6], for $p > 1$ we define $A_p^{\rho,\infty} = \bigcup_{\theta\geq 0} A_p^{\rho,\theta}$ where $A_p^{\rho,\theta}$ is the set of weights w satisfying (4).

We also introduce for $p > 1$, localized classes $A_p^{\rho,\mathrm{loc}}$ as the weights w that satisfy

$$w(B)^{1/p}[w^{-1/(p-1)}(B)]^{1/p'} \leq C|B|, \tag{13}$$

for all $B = B(x_0, r_0)$ such that $r_0 \leq \rho(x_0)$. Notice that ${}_p^{\rho,\infty} \subset_p^{\rho,\mathrm{loc}}$ for any $p > 1$. As usual, we set $A_\infty^{\rho,\infty} = \bigcup_{p>1} A_p^{\rho,\infty}$ and $A_\infty^{\rho,\mathrm{loc}} = \bigcup_{p>1} A_p^{\rho,\mathrm{loc}}$.

We state now two properties of weights in $A_p^{\rho,\infty}$ that will be needed in what follows. The first one is a self-improvement property that can be found as Lemma 5 in [6]. The second one is a doubling property and we refer the reader to Lemma 2.5 in [3].

Lemma 1. *(a) If $w \in A_p^{\rho,\infty}$, $1 < p < \infty$, then there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}^{\rho,\infty}$. Consequently*

$$\bigcup_{1<r<p} A_{p/r}^{\rho,\infty} = A_p^{\rho,\infty}. \tag{14}$$

(b) Let $w \in A_p^{\rho,\theta}$, $1 < p < \infty$, $\theta \geq 0$. If $r < R$, there exists a constant C such that for any $0 < r < R$ and $x \in \mathbb{R}^d$ the following inequality holds

$$w(B(x,R)) \leq Cw(B(x,r)) \left(\frac{R}{r}\right)^{dp} \left(1 + \frac{R}{\rho(x)}\right)^{\theta p}. \tag{15}$$

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We will also consider some maximal operators related to a critical radius function ρ. Given $\sigma > 0$ and a Young function ϕ we define

$$M_\phi^{\rho,\sigma} f(x) = \sup_{B(x_0,r_0)\ni x} \|f\|_{\phi,B(x_0,r_0)} \left(1 + \frac{r_0}{\rho(x_0)}\right)^{-\sigma}, \tag{16}$$

and

$$M_\phi^{\rho,\mathrm{loc}} f(x) = \sup_{\substack{B(x_0,r_0)\ni x \\ r_0\le\rho(x_0)}} \|f\|_{\phi,B(x_0,r_0)}, \tag{17}$$

where, as usual,

$$\|f\|_{\phi,B} = \inf\left\{\lambda > 0 : \fint_B \phi\left(\frac{|f|}{\lambda}\right) \le 1\right\}. \tag{18}$$

Whenever the function $\phi(t) = t^r$, $r \ge 1$, we will simply denote $M_r^{\rho,\sigma}$ its associated maximal. Further, if $r = 1$ we just write $M^{\rho,\sigma}$. The relation between the latter maximal operators and the classes of weights $A_p^{\rho,\infty}$ is stated in the next result.

Proposition 1. *Let $p > r \ge 1$. A weight $w \in A_{p/r}^{\rho,\infty}$ if and only if there exists $\sigma > 0$ such that $M_r^{\rho,\sigma}$ is bounded on $L^p(w)$.*

Proof. For $r = 1$ we refer to Proposition 3 in [1]. If $r > 1$ and $w \in A_{p/r}^{\rho,\infty}$ there exists $\sigma > 0$ such that $M^{\rho,\sigma}$ is bounded on $L^{p/r}(w)$. Then, for this σ,

$$\begin{aligned}\int [M_r^{\rho,\sigma} f(x)]^p w(x)dx &= \int [M^{\rho,\sigma} f^r(x)]^{p/r} w(x)dx \\ &\le C \int |f(x)|^p w(x)dx\end{aligned}$$

□

Now we turn our attention to the linear operators defined in the previous section. Boundedness of Schrödinger type singular integrals on weighted Lebesgue spaces was studied in several previous works (see, for example [6], [1] and [4]). We transcribe here results stated in Theorem 1 and Corollary 1 of [4], since they sum up boundedness properties for the precise class of operators defined above.

Theorem 3 (see Theorem 1 and Corollary 1 in [4])**.** *Let $1 < s \le \infty$ and $0 < \delta \le 1$. If T is a ρ-Schrödinger-Calderón-Zygmund operator of type (s, δ), then T is bounded on $L^p(w)$ for $s' < p < \infty$ and every $w \in A_{p/s'}^{\rho,\infty}$. Consequently, its adjoint operator is bounded on $L^p(w)$ for $1 < p < s$ and every w such that $w^{1-p'} \in A_{p'/s'}^{\rho,\infty}$ and it is of weak type $(1,1)$ with respect to w as long as $w^{s'} \in A_1^{\rho,\infty}$.*

Theorem 1 will follow essentially from the following result that can be found as Theorem 7 in [2].

Theorem 4. *Let $0 < p < \infty$, $w \in A_\infty^{\rho,\mathrm{loc}}$ and $b \in \mathrm{BMO}_\rho^\theta$. Suppose T is an integral operator of weak type (s', s') for some $1 < s < \infty$ with associated kernel K satisfying* (8) *and that for every $N \ge 0$ there exists C_N such that*

$$\sum_{k\ge1} k(2^k r)^{d/s'} \left(1 + \frac{2^k r}{\rho(x_0)}\right)^N \left(\int_{2^{k+1}B\setminus 2^k B} |K(x,y) - K(x_0,y)|^s dy\right)^{1/s} \le C, \tag{19}$$

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a.e. $x \in B$, *for every ball* $B = B(x_0, r)$ *with* $r \leq \rho(x_0)$. *Then, for any* $\sigma > 0$, *there exists a constant* C *such that*

$$\int_{\mathbb{R}^d} |[T,b]f(x)|^p w(x)dx \leq C\|b\|_{\mathrm{BMO}_\rho^\theta} \int_{\mathbb{R}^d} |M_\psi^{\rho,\sigma} f(x)|^p w(x)dx, \tag{20}$$

for every f *bounded and with compact support, where* $\psi(t) = t^{s'} \log(1+t)^{s'}$.

Let us point out that the dependence on $\|b\|_{\mathrm{BMO}_\rho^\theta}$ is not explicit in the given reference, nevertheless it can be traced back to Proposition 6 of the same work.

We give now a proof of Theorem 1.

Proof of Theorem 1. Let T be a ρ-Schrödinger-Calderón-Zygmund operator of type (s, δ) for some $1 < s < \infty$. To apply Theorem 4 we only need to show that the kernel K associated to T satisfies (19). Applying (9) together with Remark 1, we have for $0 < \delta' < \delta$,

$$\begin{aligned}\sum_{k\geq 1} k \left(2^k r\right)^{d/s'} \left(1 + \frac{2^k r}{\rho(x_0)}\right)^N & \left(\int_{2^{k+1}B\setminus 2^k B} |K(x,y) - K(x_0,y)|^s dy\right)^{1/s} \\ &\leq C \sum_{k\geq 1} k \left(2^k r\right)^{d/s'+d/s-d} 2^{-k\delta'} \\ &\leq C \sum_{k\geq 1} k 2^{-k\delta'} \leq C.\end{aligned}$$

Now, let $p > s'$, $w \in A_{p/s'}^{\rho,\infty}$ and $b \in \mathrm{BMO}_\rho^\infty$. By part (a) of Lemma 1, there exists $r \in (s', p)$ such that $w \in A_{p/r}^{\rho,\infty}$. Using Proposition 1, there exists $\sigma > 0$ such that $M_r^{\rho,\sigma}$ is bounded on $L^p(w)$. Since $b \in \mathrm{BMO}_\rho^\infty$ implies $b \in \mathrm{BMO}_\rho^\theta$ for some $\theta \geq 0$, applying Theorem 4 with such σ and θ, together with the obvious inequality $M_\psi^{\rho,\sigma} f(x) \leq M_r^{\rho,\sigma} f(x)$ it follows

$$\begin{aligned}\int_{\mathbb{R}^d} |[T,b]f(x)|^p w(x)dx &\leq C\|b\|_{\mathrm{BMO}_\rho^\theta} \int_{\mathbb{R}^d} [M_\psi^{\rho,\sigma} f(x)]^p w(x)dx \\ &\leq C\|b\|_{\mathrm{BMO}_\rho^\theta} \int_{\mathbb{R}^d} [M_r^{\rho,\sigma} f(x)]^p w(x)dx \\ &\leq C\|b\|_{\mathrm{BMO}_\rho^\theta} \int_{\mathbb{R}^d} |f(x)|^p w(x)dx.\end{aligned}$$

The case $s = \infty$ is a simple consequence of applying what has been proved for $s < \infty$ and equality (14).

□

3. Compactness for commutators of ρ-Schrödinger-Calderón-Zygmund Operators

In this section we will give a proof of Theorem 2. We will apply the following version of the Frechet-Kolmogorov criterion for compactness of subsets of $L^p(w)$ that can be found as Lemma 4.1 in [21].

Proposition 2. *Let* $1 < p < \infty$. *Let* w *be a weight on* $\mathbb{R}^d$ *such that* $w^{-p'/p}$ *is also a weight on* $\mathbb{R}^d$. *Let* G *be a subset of* $L^p(w)$. *Then* G *is relatively compact in* $L^p(w)$ *if it satisfies the following three conditions:*

(a) There exists $K > 0$ *such that* $\|f\|_{L^p(w)} \leq K$ *for all* $f \in G$.

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(b) For any $\varepsilon > 0$ there exists $R > 0$ such that

$$\left(\int_{|x|>R} |f(x)|^p w(x) dx \right)^{1/p} < \varepsilon, \text{ for any } f \in G.$$

(c) For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left(\int_{\mathbb{R}^d} |f(x+h) - f(x)|^p w(x) dx \right)^{1/p} < \varepsilon, \text{ for any } f \in G \text{ and } |h| < \delta.$$

Remark 3. Notice that $w \in A_p^{\rho,\infty}$ is a sufficient condition to apply last proposition.

Before giving a proof of Theorem 2, we make two reductions. First we show that it is enough to consider symbols $b \in \mathcal{C}_0^\infty$. Given $b \in \mathrm{CMO}_\rho^\infty$, by definition, there is $\theta \geq 0$ and a sequence of functions $\{b_n\}_{n\in\mathbb{N}} \subset \mathcal{C}_0^\infty$ that approximates b in the BMO_ρ^θ norm. Therefore, if T is a ρ-Schrödinger-Calderón-Zygmund operator of type (s,δ), $p > s'$ and $w \in A_{p/s'}^{\rho,\infty}$, by Theorem 1,

$$\|T_b f - T_{b_j} f\|_{L^p(w)} = \|T_{b-b_j} f\|_{L^p(w)} \leq C \|b - b_j\|_{\mathrm{BMO}_\rho^\theta} \|f\|_{L^p(w)}. \tag{21}$$

Then commutators with smooth symbols T_{b_j} converge to T_b as operators on $L^p(w)$. Since limit of compact operators is compact it suffices to prove compactness for commutators assuming $b \in \mathcal{C}_0^\infty$.

Given an operator T with kernel K and a number $\eta > 0$, we will denote T^η the truncated operator

$$T^\eta f(x) = \int_{|x-y|>\eta} K(x,y) f(y) dy, \tag{22}$$

and $T_{\max}^\rho$ the maximal integral operator associated to the critical radius function ρ

$$T_{\max}^\rho f(x) = \sup_{0<\eta\leq\rho(x)} |T^\eta f(x)|. \tag{23}$$

Now, following the ideas of [9], we prove the next result that will allow us to analyse only compactness of T_b^η, i.e. commutators of T^η with $\mathcal{C}_0^\infty$ symbols.

Lemma 2. *Let T be a ρ-Schrödinger-Calderón-Zygmund Operator of type (s,δ) with $1 < s < \infty$ and $\delta > 0$ and $b \in \mathcal{C}_0^\infty$. Then, for every $\sigma \geq 0$ there exists a constant C_σ such that*

$$|T_b f(x) - T_b^\eta f(x)| \leq C_\sigma \eta M_{s'}^{\rho,\sigma} f(x), \tag{24}$$

holds for almost every $x \in \mathbb{R}^d$. As a consequence T_b^η is bounded on $L^p(w)$ and $\lim_{\eta\to 0} \|T_b - T_b^\eta\|_{L^p(w)\to L^p(w)} = 0$ whenever $s' < p < \infty$ and $w \in A_{p/s'}^{\rho,\infty}$.

Proof. Let T be a ρ-Schrödinger-Calderón-Zygmund Operator of type (s,δ) for $1 < s < \infty$ and $\delta > 0$, $b \in \mathcal{C}_0^\infty$ and $\sigma \geq 0$ fixed. Using the smoothness of b, splitting in annulus and applying the size estimate (8), we obtain, for any $f \in L^p(w)$ and

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almost every $x \in \mathbb{R}^d$,

$$
\begin{aligned}
&|T_b f(x) - T_b^\eta f(x)| \\
&\quad \le \int_{|x-y|\le\eta} |b(x)-b(y)||K(x,y)||f(y)|dy \\
&\quad \le \|\nabla b\|_\infty \int_{|x-y|\le\eta} |x-y||K(x,y)||f(y)|dy \\
&\quad \le \|\nabla b\|_\infty \eta \sum_{j=0}^{\infty} 2^{-j} \left(\int_{2^{-j-1}\eta<|x-y|\le 2^{-j}\eta} |K(x,y)|^s dy \right)^{1/s} \left(\int_{|x-y|\le 2^{-j}\eta} |f(y)|^{s'} dy \right)^{1/s'} \\
&\quad \le C_N \|\nabla b\|_\infty \eta \sum_{j=0}^{\infty} 2^{-j} \left(1 + \frac{2^{-j}\eta}{\rho(x)} \right)^{-N} \left(\fint_{|x-y|\le 2^{-j}\eta} |f(y)|^{s'} dy \right)^{1/s'} \\
&\quad \le C_\sigma \|\nabla b\|_\infty \eta M_{s'}^{\sigma} f(x) \sum_{j=0}^{\infty} 2^{-j} \le C_\sigma \eta M_{s'}^{\rho,\sigma} f(x),
\end{aligned}
$$

choosing $N = \sigma$. Then, inequality (24) holds for any $\sigma \ge 0$. Next, by Proposition 1, we may select $\sigma \ge 0$ such that the maximal on the right hand side is bounded on $L^p(w)$. In this way, the remaining assertions also follow.

□

Finally, we state and prove the following boundedness result for $T^\rho_{\max}$ that is the last tool we need in order to prove Theorem 2. As expected, boundedness properties for $T^\rho_{\max}$ are similar as those obtained for T.

Theorem 5. *Let T be a ρ-Schrödinger-Calderón-Zygmund operator of type (s,δ) for $1 < s \le \infty$ and $\delta > 0$. For $s' < p < \infty$, the operator $T^\rho_{\max}$ is bounded on $L^p(w)$ as long as $w \in A^{\rho,\infty}_{p/s'}$.*

Proof. We are going to analyse first the case $s < \infty$. Let $p > s'$, $f \in L^p(w)$ and $w \in A^{\rho,\infty}_{p/s'}$. As in the classical case, we will prove first the following Cotlar type inequality: for each $\nu > s'$ and $\sigma \ge 0$ there is a constant C such that for almost every $x \in \mathbb{R}^d$.

$$
T^\rho_{\max} f(x) \le C \left[M^{\rho,\sigma}_{s'} f(x) + M^{\rho,\mathrm{loc}}_{\nu} f(x) + M^{\rho,\mathrm{loc}}(Tf)(x) \right]. \tag{25}
$$

We first check that $T^\varepsilon f(x)$ is finite for almost every $x \in \mathbb{R}^d$. Applying estimate (8) we obtain, for any $\sigma \ge 0$,

$$
\begin{aligned}
&\int_{|x-y|>\varepsilon} |K(x,y)||f(y)|dy \\
&\qquad \le \sum_{j=1}^{\infty} \left(\int_{2^{j-1}\varepsilon<|x-y|\le 2^j\varepsilon} |K(x,y)|^s \right)^{1/s} \left(\int_{|x-y|\le 2^j\varepsilon} |f(y)|^{s'} \right)^{1/s'} \\
&\qquad \le C_N \sum_{j=1}^{\infty} \left(1 + \frac{2^k\varepsilon}{\rho(x)} \right)^{-N} (2^k\varepsilon)^{-d/s'} \left(\int_{|x-y|\le 2^j\varepsilon} |f(y)|^{s'} \right)^{1/s'} \\
&\qquad \le C_N M^{\rho,\sigma}_{s'} f(x) \sum_{j=1}^{\infty} \left(1 + \frac{2^k\varepsilon}{\rho(x)} \right)^{-N+\sigma}.
\end{aligned} \tag{26}
$$

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By Proposition 1, there exists $\sigma \geq 0$ such that $M_{s'}^{\rho,\sigma}$ is bounded on $L^p(w)$. Choosing that σ and $N = \sigma + 1$

$$\int_{|x-y|>\varepsilon} |K(x,y)||f(y)|dy \leq C_\sigma M_{s'}^{\rho,\sigma} f(x) \sum_{j=1}^{\infty} \frac{\rho(x)}{2^k \varepsilon} \leq C(\sigma,\varepsilon,x) M_{s'}^{\rho,\sigma} f(x), \tag{27}$$

where $C(\sigma, \varepsilon, x)$ is a finite number for any x. Therefore, for any $\varepsilon > 0$, $T^\varepsilon f(x)$ is finite as long as $M_{s'}^{\rho,\sigma} f(x)$ is so and by the choice of σ this happens for almost every $x \in \mathbb{R}^d$.

Set $f_1^x = f\chi_{B(x,\varepsilon)}$ and $f_2^x = f\chi_{B(x,\varepsilon)^c}$. Notice that

$$T f_2^x(x) = \int K(x,y) f_2^x(y) = \int_{|x-y|\geq\varepsilon} K(x,y) f(y)dy = T^\varepsilon f(x).$$

Observe that for almost every $z \in \mathbb{R}^d$, $T f_2^x(z)$ is finite. So, for $z \in B(x, \varepsilon/2)$, we apply the smoothness estimate (10) to obtain

$$\begin{aligned}
|T f_2^x(x) - T f_2^x(z)| &= \left| \int_{|x-y|\geq\varepsilon} [K(x,y) - K(z,y)] f(y)dy \right| \\
&\leq \sum_{j=1}^{\infty} \left(\int_{2^{j-1}\varepsilon \leq |x-y| < 2^j\varepsilon} |K(x,y) - K(z,y)|^s dy \right)^{1/s} \left(\int_{|x-y|<2^j\varepsilon} |f(y)|^{s'} dy \right)^{1/s'} \\
&\leq C_N \sum_{j=1}^{\infty} 2^{-j\delta'} \left(1 + \frac{2^j\varepsilon}{\rho(x)}\right)^{-N} \left(\fint_{|x-y|<2^j\varepsilon} |f(y)|^{s'} dy \right)^{1/s'} \\
&\leq C_\sigma M_{s'}^{\rho,\sigma} f(x),
\end{aligned}$$

choosing $N = \sigma$ and $0 < \delta' < \delta$.

Therefore, for $z \in B(x, \varepsilon/2)$,

$$\begin{aligned}
|T^\varepsilon f(x)| &= |T f_2^x(x)| \\
&\leq |T f_2^x(x) - T f_2^x(z)| + |T f_2^x(z)| \\
&\leq C M_{s'}^{\rho,\sigma} f(x) + |T f_1^x(z)| + |T f(z)|.
\end{aligned}$$

Now, taking average over $B(x, \varepsilon/2)$ we obtain

$$|T^\varepsilon f(x)| \leq C M_{s'}^{\rho,\sigma} f(x) + \fint_{B(x,\varepsilon/2)} |T f_1^x(z)|dz + M^{\rho,\mathrm{loc}}(Tf)(x).$$

To bound the second term of the last expression we can pick $\nu \in (s', p)$ and use the boundedness of T on L^ν to obtain

$$\begin{aligned}
\fint_{B(x,\varepsilon/2)} |T f_1^x(z)|dz &\leq \left(\fint_{B(x,\varepsilon/2)} |T f_1^x(z)|^\nu dz \right)^{1/\nu} \\
&\leq C \left(\frac{1}{|B(x,\varepsilon/2)|} \int_{\mathbb{R}^d} |f_1^x(z)|^\nu dz \right)^{1/\nu} \\
&\leq C \left(\frac{1}{|B(x,\varepsilon/2)|} \int_{B(x,\varepsilon)} |f(z)|^\nu dz \right)^{1/\nu} \\
&\leq C M_\nu^{\rho,\mathrm{loc}} f(x),
\end{aligned}$$

leading to (25) once we have taken the supreme on $0 < \varepsilon \leq \rho(x)$.

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Now, given $p > s'$ and $w \in A^{\rho,\infty}_{p/s'}$ we apply Lemma 1 to obtain $\nu \in (s', p)$ such that $w \in A^{\rho,\infty}_{p/\nu}$. Then, by Proposition 1, there exists $\sigma > 0$ such that $M^{\rho,\sigma}_{\nu}$ is bounded on $L^p(w)$. Applying (25) for such ν and σ,

$$\begin{aligned} T^{\rho}_{\max} f(x) &\leq C\left[M^{\rho,\sigma}_{s'} f(x) + M^{\rho,\mathrm{loc}}_{\nu} f(x) + M^{\rho,\mathrm{loc}}(Tf)(x)\right] \\ &\leq C\left[M^{\rho,\sigma}_{\nu} f(x) + M^{\rho,\sigma}_{\nu}(Tf)(x)\right]. \end{aligned}$$

Finally, the boundedness of $T^{\rho}_{\max}$ on $L^p(w)$ follows from the boundedness on $L^p(w)$ of $M^{\rho,\sigma}_{\nu}$ and T, the last one provided by Theorem 3. If $s = \infty$, the result follows from the case of s finite we just proved together with part (a) of Lemma 1 as was done in the proof of Theorem 1. □

Now we have all set up to proceed with the proof of the main result of this section.

Proof of Theorem 2. We will consider first the case $s < \infty$. Let $s' < p < \infty$, $b \in \mathcal{C}^{\infty}_0$ and $w \in A^{\rho,\infty}_{p/s'}$. Since the limit in operator norm of compact operators is compact and in view of Lemma 2, it suffices to prove compactness for T^{η}_b on $L^p(w)$ for $\eta > 0$ and small enough.

Notice that we can apply Proposition 2 since in particular $w \in A^{\rho,\infty}_p$ (see Remark 3). Consider $G = \{T^{\eta}_b f : \|f\|_{L^p(w)} \leq 1\}$. Condition (a) of Proposition 2 is satisfied since T^{η}_b is a bounded operator on $L^p(w)$ according to Lemma 2.

To check condition (b) of Proposition 2 we consider R_0 such that $\operatorname{supp} b \subset B_0 = B(0, R_0)$ and x such that $|x| > 2R_0$, and then $b(x) = 0$. Then, if $y \in \operatorname{supp} b$, $|x|/2 \leq |x - y| \leq 3|x|/2$ and

$$\begin{aligned} |T^{\eta}_b f(x)| &= \left|\int [b(x) - b(y)] K^{\eta}(x, y) f(y) dy\right| \\ &\leq \|b\|_{\infty} \int_{\operatorname{supp} b} |K(x, y)||f(y)| w^{1/p}(y) w^{-1/p}(y) dy \\ &\leq \|b\|_{\infty} \left(\int_{B_0} |K(x, y)|^s dy\right)^{1/s} \left(\int_{B_0} |f(y)|^p w(y) dy\right)^{1/p} \\ &\quad \times \left(\int_{B_0} w^{-\gamma/p}(y) dy\right)^{1/\gamma}, \end{aligned}$$

where $\gamma > 1$ is such that $s^{-1} + p^{-1} + \gamma^{-1} = 1$. Simple calculations show that $w \in A^{\rho,\infty}_{p/s'}$ implies $w^{-\gamma/p} \in L^1_{\mathrm{loc}}$. Then, applying the size estimate (8) we obtain

$$\begin{aligned} |T^{\eta}_b f(x)| &\leq C\|b\|_{\infty}\|f\|_{L^p(w)} \left(\int_{|x|/2 \leq |x-y| \leq 3|x|/2} |K(x, y)|^s dy\right)^{1/s} \\ &\leq \frac{C_N}{|x|^{d/s'}} \left(1 + \frac{|x|}{\rho(x)}\right)^{-N} \\ &\leq \frac{C_N}{|x|^{d/s'}} \left(1 + \frac{|x|}{\rho(0)}\right)^{\frac{-N}{N_0+1}}, \end{aligned}$$

where the last inequality follows from the right hand side inequality given in (3).

By Lemma 1 there exists $q < p$ such that $w \in A^{\rho,\infty}_{q/s'}$ which implies $w \in A^{\rho,\theta}_{q/s'}$ for some $\theta \geq 0$. Using this fact and the doubling property stated also in Lemma 1 we

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obtain that for $R > 2R_0$,

$$\begin{aligned}\int_{|x|>R} |T_b^\eta f(x)|^p w(x)dx &\le C_N \int_{|x|>R} \left(1+\frac{|x|}{\rho(0)}\right)^{\frac{-Np}{N_0+1}} \frac{w(x)}{|x|^{dp/s'}}dx \\ &\le C_N \sum_{j=1}^{\infty} (2^j R)^{-dp/s'} \int_{2^{j-1}R<|x|\le 2^j R} \left(1+\frac{|x|}{\rho(0)}\right)^{\frac{-Np}{N_0+1}} w(x)dx \\ &\le C_N \sum_{j=1}^{\infty} (2^j R)^{-dp/s'} \left(1+\frac{2^j R}{\rho(0)}\right)^{\frac{-Np}{N_0+1}} w(B(0,2^j R)) \\ &\le C_N \sum_{j=1}^{\infty} (2^j R)^{\frac{-dp+dq}{s'}} \left(1+\frac{2^j R}{\rho(0)}\right)^{\frac{-Np}{N_0+1}+\frac{\theta q}{s'}} w(B(0,1)) \\ &\le C R^{\frac{d(q-p)}{s'}} \sum_{j=1}^{\infty} 2^{\frac{jd(q-p)}{s'}},\end{aligned}$$

choosing $N > \theta q(N_0+1)/(ps')$. Since $q < p$, the series is convergent and the last term goes to 0 as R goes to infinity as we wanted to show.

Finally, we are going to check condition (c) of Proposition (2). In order to do that we write

$$\begin{aligned}T_b^\eta f(x) - T_b^\eta f(x+h) &= \int [b(x)-b(y)]K^\eta(x,y)f(y)dy \\ &\quad - \int [b(x+h)-b(y)]K^\eta(x+h,y)f(y)dy \\ &= [b(x)-b(x+h)] \int K^\eta(x,y)f(y)dy \\ &\quad + \int [b(x+h)-b(y)][K^\eta(x,y)-K^\eta(x+h,y)]f(y)dy \\ &= A + B.\end{aligned}$$

First, we notice that $A = 0$ if $x \notin 2B_0$ since we may suppose $|h| < R_0$. Set $\rho_0 = \inf_{z\in 2B_0} \rho(z)$. That $\rho_0 > 0$ follows from the left hand side of (3). In fact, taking there $y = z$ and $x = 0$ we obtain

$$\rho(z) \ge c_\rho^{-1}\rho(0)\left(1+\frac{|z|}{\rho(0)}\right)^{-N_0} \ge c_\rho^{-1}\rho(0)\left(1+\frac{2R_0}{\rho(0)}\right)^{-N},$$

whenever $|z| < 2R_0$. Then, going back to A, for $\eta < \rho_0$ and $|x| > 2R_0$, we obtain

$$|A| = |b(x)-b(x+h)| \left|\int_{|x-y|>\eta} K(x,y)f(y)dy\right| \le \|\nabla b\|_\infty |h| T^\rho_{\max} f(x).$$

By Theorem 5, $T^\rho_{\max}$ is bounded on $L^p(w)$ and then

$$\|A\|_{L^p(w)} \le C|h|\|f\|_{L^p(w)},$$

and the last term goes to zero with $|h|$ as we wanted to show.

To deal with B we first note that the integrand is zero if $|x+h-y| < \eta$ and $|x-y| < \eta$, so we only need to integrate over $I = [B(x,\eta) \cap B(x+h,\eta)]^c$. To this

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end we suppose $|h| < \eta/2$ and decompose $I = I_1 \cup (I \setminus I_1)$ where

$$I_1 = \{y \in \mathbb{R}^d : |x-y| > \eta, |x+h-y| > \eta\}$$

and we observe that

$$I \setminus I_1 \subset I_2 = \{y \in \mathbb{R}^d : \eta - |h| < |x-y| < \eta + |h|\}.$$

Accordingly, we decompose B in two terms, B_1 and B_2, integrating over I_1 and $I \setminus I_1$ respectively. For the first term we can suppose that $|h|$ is small enough to use the smoothness estimate for K given in (10) (in fact $|h| < \eta/2$ will work). Also, since $w \in A^{\rho,\infty}_{p/s'}$ we know there exists $\sigma > 0$ such that $M^{\rho,\sigma}_{s'}$ is bounded on $L^p(w)$ for $p > s'$ by Proposition 1. Then

$$\begin{aligned}
|B_1| &\leq \int_{I_1} |b(x+h) - b(y)||K(x,y) - K(x+h,y)||f(y)|dy \\
&\leq 2\|b\|_\infty \int_{|x-y|>\eta} |K(x,y) - K(x+h,y)||f(y)|dy \\
&\leq 2\|b\|_\infty \sum_{j=1}^{\infty} \left(\int_{2^{j-1}\eta<|x-y|<2^j\eta} |K(x,y) - K(x+h,y)|^s dy \right)^{1/s} \\
&\quad \times \left(\int_{|x-y|<2^j\eta} |f(y)|^{s'} dy \right)^{1/s'} \\
&\leq C_N \|b\|_\infty \sum_{j=0}^{\infty} \left(\frac{|h|}{2^j\eta} \right)^{\delta'} \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-N} \left(⨍_{|x-y|<2^j\eta} |f(y)|^{s'} dy \right)^{1/s'} \\
&\leq C_N \|b\|_\infty \sum_{j=0}^{\infty} \left(\frac{|h|}{2^j\eta} \right)^{\delta'} \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-N} \left(⨍_{|x-y|<2^j\eta} |f(y)|^{s'} dy \right)^{1/s'} \\
&\leq C_\sigma \left(\frac{|h|}{\eta} \right)^{\delta'} \|b\|_\infty M^{\rho,\sigma}_{s'} f(x) \sum_{j=1}^{\infty} 2^{-j\delta'} \leq C|h|^{\delta'} M^{\rho,\sigma}_{s'} f(x).
\end{aligned}$$

Here, we have chosen $N = \sigma$ and $0 < \delta' < \delta$.

Now, we may apply the boundedness of $M^{\rho,\sigma}_{s'}$ on $L^p(w)$ for $p > s'$ to obtain,

$$\|B_1\|_{L^p(w)} \leq C|h|^{\delta'} \|f\|_{L^p(w)},$$

and the last term goes to zero with $|h|$ as we wanted to show.

To deal with the second term, we pick as before R_0 such that $\operatorname{supp} b \subset B(0, R_0)$. Choosing h and η such that $|h| < \eta/4 < R_0$ we have $\operatorname{supp} b(\cdot + h) \subset B(0, 2R_0)$. If $|x| > 4R_0$, $b(x+h) = 0$ and $b(y) = 0$ since for $y \in I_2$,

$$|y| \geq |x| - |x-y| > 4R_0 - 3\eta/2 > 4R_0 - 3R_0 = R_0.$$

Hence, to estimate B_2 it is enough to take care of $|x| < 4R_0$. In this case we observe that

$$I_2 \subset B(x, 3\eta/2) \setminus B(x, \eta/2)$$

since $\eta + |h| < 3\eta/2$ and $\eta - |h| > \eta/2$. Also,

$$B(x, 3\eta/2) \subset B(0, 7R_0)$$

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since $|y| \leq |x-y| + |x| \leq 3\eta/2 + |x| \leq 3R_0 + 4R_0 = 7R_0$.

Hence, for $|x| < 4R_0$,

$$|B_2| \leq 2\|b\|_\infty \left[\int_{I_2} |K(x+h,y)||f(y)|dy + \int_{I_2} |K(x,y)||f(y)|dy\right] = 2\|b\|_\infty [B_{21} + B_{22}].$$

To bound the first integral, we apply Hölder's inequality to obtain

$$B_{21} \leq \left(\int_{I_2} |K(x+h,y)|^s dy\right) \left(\int |f|^p w\right)^{1/p} \left[w^{-\gamma/p}(I_2)\right]^{1/\gamma},$$

where γ is such that $s^{-1} + p^{-1} + \gamma^{-1} = 1$ as above. Now, since $I_2 \subset B(x, 3\eta/2) \setminus B(x, \eta/2)$, using (8), we get

$$B_{21} \leq C\eta^{-d/s'} \|f\|_{L^p(w)} \left[w^{-\gamma/p}(I_2)\right]^{1/\gamma}.$$

Therefore, since η is fixed we are led to

$$\|B_{21}\|^p_{L^p(w)} \leq C \int_{|x|<4R_0} \left[w^{-\gamma/p}(I_2)\right]^{p/\gamma} w(x)dx = C \int_{|x|<4R_0} g_h(x)w(x)dx,$$

where $g_h(x) = \left[w^{-\gamma/p}(I_2)\right]^{p/\gamma}$ since I_2 depends on h and x. Since $w \in A^{\rho,\infty}_{p/s'}$ we have that $w^{-\gamma/p} \in L^1_{\text{loc}}$ and then, for each x, $g_h(x) \to 0$ with h. Also, since $I_2 \subset B(0, 7R_0)$ we have

$$g_h(x) \leq \left[w^{-\gamma/p}(B(0, 7R_0))\right]^{p/\gamma} \leq C.$$

Therefore, we can apply the Dominated Convergence Theorem to conclude

$$\|B_{21}\|^p_{L^p(w)} \leq C \int_{|x|<4R_0} g_h(x)w(x)dx \to 0,$$

when $|h| \to 0$, as we wanted to show. To deal with B_{22} we proceed in a similar way.

Altogether, we have proved that for small values of η the operator T^η_b satisfies the three conditions required by the compactness criterion given in Proposition 2 as long as $s < \infty$. Finally, to deal with the case $s = \infty$ we follow the same procedure as in the proof of Theorem 1.

□

4. Applications

In this section we will apply Theorem 1 and Theorem 2 to obtain continuity and compactness on $L^p(w)$ for commutators of several examples of operators related to $L = -\Delta + V$ with potentials satisfying a reverse Hölder condition as stated in the introduction. Therefore, from now on, ρ will be the critical radius function associated to the potential by means of (2).

The operators to be considered here are the following:

- The first order Schrödinger-Riesz Transform $\mathcal{R}_1 = \nabla L^{-1/2}$,
- the second order Schrödinger-Riesz Transform $\mathcal{R}_2 = \nabla^2 L^{-1}$,
- the family of operators $T_\gamma = V^\gamma L^{-\gamma}$ for $0 < \gamma < d/2$ and
- the family of operators $S_\gamma = V^{\gamma-1/2}\nabla L^{-\gamma}$ for $1/2 < \gamma \leq 1$,
- the family of operators $L^{i\alpha}$ for $\alpha \in \mathbb{R}$.

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The Schrödinger-Riesz transforms were first studied by Shen in [17], together with the operators T_γ for $\gamma = 1/2$ and $\gamma = 1$, S_γ for $\gamma = 1$, and $L^{i\alpha}$ for $\alpha \in \mathbb{R}$. There, he proved the L^p boundedness of these operators for p in an interval of the form $(1, s]$ or $(1, \infty)$.

It is important noting that the class of operators considered in Theorem 1 and Theorem 2, models, in most cases, not the operators listed above but their adjoints. Therefore, many times, we are going to show that the adjoint operators of the mentioned examples fit into these classes.

In the following proposition we summarize the results obtained in [4], where the Schrödinger-Calderón-Zygmund classes of operators are widely discussed. However, it is worth mentioning that some of the estimates given there can be traced back to [17] and [7]. For shortness we will use the notation SCZ to mean a ρ-Schrödinger-Calderón-Zygmund operator when ρ is the critical radius function derived from the potential V.

Proposition 3. *Let $V \geq 0$ satisfying a reverse Hölder inequality of order $q > d/2$ with $d > 2$ Then, for some $\delta > 0$, which may be different at each occurrence, we have:*

(i) $\mathcal{R}_1$ and $\mathcal{R}_1^*$ are SCZ of type (∞, δ) if we further ask $q \geq d$.
(ii) $\mathcal{R}_1^*$ is a SCZ of type (p_0, δ), with p_0 such that $1/p_0 = 1/q - 1/d$.
(iii) $\mathcal{R}_2^*$ is a SCZ of type (q, δ).
(iv) $L^{-\gamma}V^\gamma$ is a SCZ of type (q/γ) for $0 < \gamma < d/2$.
(v) $L^{-\gamma}\nabla V^{\gamma-1/2}$ is a SCZ of type (q_γ, δ) where q_γ is such that

$$1/q_\gamma = (1/q - 1/d)^+ + (2\gamma - 1)/2q \tag{28}$$

with $1/2 < \gamma \leq 1$.

Proof. In [17], Shen proves that $\mathcal{R}_1$ is a Calderón-Zygmund operator. From that it follows the continuity in all L^p and the smoothness of the kernel in each variable. The size condition with the extra decay also appears there as inequality (6.5). So, $\mathcal{R}_1$ and $\mathcal{R}_1^*$ are of type (∞, δ).

To prove (ii), we recall that strong boundedness on $L^{p_0'}$ was proved in [17]. The required kernel conditions can be found in [7] (see Lemma 6 and Lemma 7 there).

As for (iii), we observe that Theorem 0.3 in [17] gives the strong type (q', q') and the remaining conditions follow from Proposition 8 in [4].

Finally, (iv) and (v) can be found as Proposition 7 and Proposition 6 from [4]. □

Regarding the operator $L^{i\alpha}$, from [17], it is easy to check that the following proposition holds.

Proposition 4. *Let $V \geq 0$ satisfying a reverse Hölder inequality of order $q > d/2$ with $d > 2$. For $\alpha \in \mathbb{R}$, $L^{i\alpha}$ is a SCZ of type (∞, δ) for some $\delta > 0$.*

Proof. The proof of this statement is contained in Theorem 0.4 of [17]. There it is proved that $L^{i\alpha}$ is a standard Calderón-Zygmund operator. Therefore, the weak type (p, p) for all $p > 1$ is guaranteed. From this fact, we also obtain the smoothness estimate (12). As for the size, we refer to equation (4.3) in [17], which shows that (11) also holds. The result follows then by Remark 2. □

Now, as a consequence of the last results, Theorem 1 and Theorem 2, we establish the continuity and compactness on $L^p(w)$ for commutators of singular integral operators associated to L.

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All along the following theorems we will assume that $V \in \mathrm{RH}_q$ with $d/2 < q < \infty$ and $b \in \mathrm{BMO}_\rho^\infty$ for the function ρ given by (2).

First, for the commutator of $\mathcal{R}_1$ we have the following result. The boundedness property stated below recovers Theorem 2 in [7]. Compactness of commutators with CMO_ρ^∞ symbols is new.

Theorem 6. *Let p_0 satisfying $1/p_0 = (1/q - 1/d)^+$. Then, for any $1 < p < p_0$ and w such that $w^{1-p'} \in A^{\rho,\infty}_{p'/p_0'}$, the commutator $[\mathcal{R}_1, b]$ is a bounded operator on $L^p(w)$. Further, if $b \in \mathrm{CMO}_\rho^\infty$, it is compact.*

In particular, if $q \geq d$, the above statements are valid for any $1 < p < \infty$ and $w \in A_p^{\rho,\infty}$.

Proof. If $q < d$, boundedness and compactness follow by a standard duality argument once we have applied Theorem 1 and Theorem 2 to $\mathcal{R}_1^\star$ which, according to Proposition 3, is a ρ-Schrödinger-Calderón-Zygmund operator of type (p_0, δ). If $q \geq d$ we can apply Theorem 1 and Theorem 2 directly to $\mathcal{R}_1$.

□

For commutators associated to the second order Schrödinger-Riesz transform $\mathcal{R}_2$, both boundedness and compactness results are new. In the next theorems it is worth noting that $V \in \mathrm{RH}_\infty$ implies $V \in \mathrm{RH}_q$ for all $q > 1$.

Theorem 7. *For any $1 < p < q$ and w such that $w^{1-p'} \in A^{\rho,\infty}_{p'/q'}$, the commutator $[\mathcal{R}_2, b]$ is a bounded operator on $L^p(w)$. Further, if $b \in \mathrm{CMO}_\rho^\infty$, it is compact.*

Moreover, if $V \in \mathrm{RH}_q$ for all $1 < q < \infty$, the above statements are valid for any $1 < p < \infty$ and $w \in A_p^{\rho,\infty}$.

Proof. If $V \in \mathrm{RH}_q$ for some $q < \infty$, the boundedness and compactness follow again by a duality argument once Theorem 1 and Theorem 2 are applied to $\mathcal{R}_2^\star$ which is a ρ-Schrödinger-Calderón-Zygmund operator of type (q, δ) according to Proposition 3. If $V \in \mathrm{RH}_q$ for all $1 < q < \infty$ we obtain boundedness and compactness for commutators of $\mathcal{R}_2^\star$ for any p, $1 < p < \infty$ and weights in $A_p^{\rho,\infty}$, according to Lemma 1. By duality, since $w^{1-p'} \in A^{\rho,\infty}_{p'}$ is equivalent to $w \in A_p^{\rho,\infty}$, the last assertion of the theorem follows.

□

Following similar arguments to the last proof we can easily derive the two following theorems concerning T_γ and S_γ. The boundedness of the commutators $[T_\gamma, b]$ and $[S_\gamma, b]$ on $L^p(w)$ with BMO_ρ^∞ symbols was obtained by Tang in [18] but only for the cases $\gamma = 1$ and $\gamma = 1/2$ for T_γ and $\gamma = 1$ for S_γ. Here we show that continuity results hold for a range of γ that includes the mentioned cases. The compactness results obtained for $[T_\gamma, b]$ and $[S_\gamma, b]$ are completely new.

Mirar este parrafo, hay otras citas pero creo que no van

Theorem 8. *Let $0 < \gamma < d/2$. Then, for any $1 < p < q/\gamma$ and w such that $w^{1-p'} \in A^{\rho,\infty}_{p'/(q/\gamma)'}$, the commutator $[T_\gamma, b]$ is a bounded operator on $L^p(w)$. Further, if $b \in \mathrm{CMO}_\rho^\infty$, it is compact.*

Moreover, if $V \in \mathrm{RH}_q$ for all $1 < q < \infty$, the above statements are valid for any $1 < p < \infty$ and $w \in A_p^{\rho,\infty}$.

Theorem 9. *Let $1/2 < \gamma \leq 1$ and q_γ as defined in (28). Then, for any $1 < p < q_\gamma$ and w such that $w^{1-p'} \in A^{\rho,\infty}_{p'/q_\gamma'}$, the commutator $[S_\gamma, b]$ is a bounded operator on $L^p(w)$. Further, if $b \in \mathrm{CMO}_\rho^\infty$, it is compact.*

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Moreover, if $V \in \mathrm{RH}_q$ for all $1 < q < \infty$, the above statements are valid for any $1 < p < \infty$ and $w \in A_p^{\rho,\infty}$.

Remark 4. We point out that even in the above theorems we obtain similar results for $\mathcal{R}_1$, $\mathcal{R}_2$, T_γ and S_γ under stronger assumptions on V, the situation is not the same in all cases. Pointwise estimates as (11) and (12) are known for the kernels of $\mathcal{R}_1$ and $\mathcal{R}_1^\star$ when $V \in \mathrm{RH}_d$. However, the situation is different for $\mathcal{R}_2^\star$, $T_\gamma^\star$ and $S_\gamma^\star$ since, to our knowledge, such estimates have not been proved.

As for the operator $L^{i\alpha}$, the following result follows straightforwardly from Theorem 1, Theorem 2 and Proposition 4. The boundedness of $[L^{i\alpha}, b]$ on $L^p(w)$ with BMO_ρ^∞ symbols was already obtained by Tang in [18]. The compactness result is new.

Theorem 10. *Let $V \geq 0$ satisfying a reverse Hölder inequality of order $q > d/2$ with $d > 2$ and $\alpha \in \mathbb{R}$. Then, for any $1 < p < \infty$ and $w \in A_p^{\rho,\infty}$, the commutator $[L^{i\alpha}, b]$ is a bounded operator on $L^p(w)$. Further, if $b \in \mathrm{CMO}_\rho^\infty$, it is compact.*

5. On a characterization of CMO_ρ

As in Uchiyama, we present here a description of our space for the symbols in terms of smallness of (ρ, θ) mean oscillations. More precisely we introduce the class SMO_ρ^∞ as those functions in BMO_ρ^∞ such that for some $\theta \geq 0$ satisfy the following three conditions

(i) $\lim_{\delta \to 0} \sup \left\{ \mathrm{Osc}_\rho^\theta(f, B(x,r)) : x \in \mathbb{R}^d, r \leq \delta \right\} = 0$

(ii) $\lim_{\lambda \to \infty} \sup \left\{ \mathrm{Osc}_\rho^\theta(f, B(x,r)) : x \in \mathbb{R}^d, r \geq \lambda \right\} = 0$

(iii) $\lim_{|x| \to \infty} \mathrm{Osc}_\rho^\theta(f, B(x,r)) = 0$, for each fixed $r > 0$.

When the above conditions are satisfied with some particular θ we say that $f \in \mathrm{SMO}_\rho^\theta$. Notice that if they hold for some θ, they are also true for any larger value. The above three conditions can be rephrased in a very useful way.

Lemma 3. *Let $f \in \mathrm{SMO}_\rho^\theta$ for some $\theta \geq 0$. Then, for each $\epsilon > 0$ there exists δ, λ and R such that*

$$\mathrm{Osc}_\rho^\theta(f, B(x,r)) < \epsilon$$

for any x provided $r < \delta$ or $r > \lambda$ and for any r as long as $|x| > R$.

Proof. According to (i) and (ii) we choose δ and λ with the desired properties. Pick now two integers i and j such that $2^i \leq \delta < 2^{i+1}$ and $2^{j-1} \leq \lambda < 2^j$. Clearly we may assume $i < j$. Now, for all $i < k < j$, according to (iii) and being a finite set of radius, we may choose R such that

$$\mathrm{Osc}_\rho^\theta(f, B(x, 2^k)) < c\,\epsilon \quad \text{for } |x| > R,$$

where $c < 1$ is a fixed constant that will be chosen later. Now we analyse what happens for any $2^i < r < 2^j$. In fact, pick k such that $2^k < r < 2^{k+1}$ with $i \leq k \leq j$. Then, since $B(x,r) \subset B(x, 2^{k+1})$ and $2^{k+1}/r < 2$, we easily obtain

$$\mathrm{Osc}_\rho^\theta(f, B(x,r)) \leq C \; \mathrm{Osc}_\rho^\theta(f, B(x, 2^{k+1})),$$

where the constant C depends only on θ and the dimension. Therefore, choosing $c = C^{-1}$ we arrive to the desired conclusion. ☐

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Observe also that for each $\theta \geq 0$, the space SMO^θ_ρ is a vector subspace of BMO^θ_ρ. Moreover it is closed with respect to the $\|\cdot\|_{\mathrm{BMO}^\theta_\rho}$ norm. In fact, suppose f is in the closure of SMO^θ_ρ and let $\epsilon > 0$ be given. Then there exists a function $g \in \mathrm{SMO}^\theta_\rho$ such that

$$\sup_{B(x,r)} \mathrm{Osc}^\theta_\rho(f-g, B(x,r)) < \epsilon/2.$$

For this function g there exist δ_0 and λ_0 such that for any x and either $r < \delta_0$ or $r > \lambda_0$,

$$\mathrm{Osc}^\theta_\rho(g, B(x,r)) < \epsilon/2.$$

Moreover, for any given r there is some $R = R(r)$ such that

$$\sup_{|x|>R} \mathrm{Osc}^\theta_\rho(g, B(x,r)) < \epsilon/2.$$

Using now

$$\mathrm{Osc}^\theta_\rho(f, B(x,r)) \leq \mathrm{Osc}^\theta_\rho(f-g, B(x,r)) + \mathrm{Osc}^\theta_\rho(g, B(x,r)),$$

it follows that f also satisfies the three required conditions to be in SMO^θ_ρ.

Another observation is that $\mathcal{C}^\infty_0$, the space of functions with infinitely many derivatives and compact support, is contained in SMO^θ_ρ. Moreover this is true for continuous functions of compact support. In fact a such function is uniformly continuous, and so, given $\epsilon > 0$ for some δ we have $|x-y| < \delta$ implies $|f(x)-f(y)| < \epsilon$. Therefore, for $r < \delta$,

$$\mathrm{Osc}^\theta_\rho(f, B(x_0,r)) \leq \mathrm{Osc}(f, B(x_0,r)) \leq \frac{1}{|B|^2}\int_B\int_B |f(x)-f(y)|dxdy < \epsilon.$$

Next, as f is integrable,

$$\mathrm{Osc}^\theta_\rho(f, B(x_0,r)) \leq 2|f_{B(x_0,r)}| \leq \frac{C}{r^d}\int_{\mathbb{R}^d} |f|,$$

and the right hand side goes to zero when r tends to infinity.

Finally, suppose that the support of f is contained in a ball $B(0,R_0)$. Then, if for a fixed r we take $|x_0| > r + R_0$, we get $f = 0$ over $B(x_0,r)$.

In view of the above properties for SMO^θ_ρ and the definition of CMO^θ_ρ it follows that $\mathrm{CMO}^\theta_\rho \subset \mathrm{SMO}^\theta_\rho$ for every $\theta \geq 0$, and so we have

$$\mathrm{CMO}^\infty_\rho \subset \mathrm{SMO}^\infty_\rho.$$

The remain of the section is devoted to analyse the other inclusion. First we observe that if $\theta_1 \leq \theta_2$ then $\mathrm{SMO}^{\theta_1}_\rho \subset \mathrm{SMO}^{\theta_2}_\rho$ and therefore we can write $\mathrm{SMO}^\infty_\rho = \cup_{\theta\geq 1}\mathrm{SMO}^\theta_\rho$. So, to show $\mathrm{SMO}^\infty_\rho \subset \mathrm{CMO}^\infty_\rho$, it will be enough to prove

$$\mathrm{SMO}^\theta_\rho \subset \mathrm{CMO}^\theta_\rho, \quad \text{for } \theta \geq 1.$$

However, we are able to do that assuming that the critical function ρ is bounded over the whole space. Even though it imposes some restriction, the characterization will hold for a large class of potentials, for instance, for positive polynomial potentials, in particular for the Hermite operator $Hu(x) = -\Delta u(x) + |x|^2 u(x)$.

We will prove the inclusion $\mathrm{SMO}^\theta_\rho \subset \mathrm{CMO}^\theta_\rho$ for bounded ρ and $\theta \geq 1$ by performing a series of approximations. First we reduce to the case $f \in \mathrm{SMO}^\theta_\rho$ and bounded, then we approximate by a bounded function with compact support lying

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in SMO_ρ^θ and finally by a $\mathcal{C}_0^\infty$ function, always with respect to the norm (6). Let us point out that only for the second step we will make use of the extra condition on ρ. First we need to prove some technical lemmas.

Lemma 4. *Let f be a function in SMO_ρ^θ and c any constant. Then $g = \min\{f, c\}$ belongs also to SMO_ρ^θ and moreover it satisfies*

$$\mathrm{Osc}_\rho^\theta(g, B(x_0, r)) \leq 2\,\mathrm{Osc}_\rho^\theta(f, B(x_0, r)),$$

for any ball $B(x_0, r)$.

Proof. First observe

$$\min\{f(x), c\} = \frac{1}{2}\left(f(x) + c - |f(x) - c|\right),$$

so that

$$\begin{aligned}|g(x) - g(y)| &\leq 1/2\left(|f(x) - f(y)| + ||f(x) + c| - |f(y) - c||\right)\\ &\leq |f(x) - f(y)|.\end{aligned}$$

Therefore, for any ball $B(x_0, r)$,

$$\int_{B(x_0,r)}\int_{B(x_0,r)} |g(x) - g(y)|\,dx\,dy \leq \int_{B(x_0,r)}\int_{B(x_0,r)} |f(x) - f(y)|\,dx\,dy,$$

and the conclusion follows for any ρ and θ. □

Lemma 5. *Let ρ be a critical radius function, and x and z two points of $\mathbb{R}^d$. Then, there exists a constant C, depending only on the constant in (3), such that if $|x - z| \leq 1$, we have $\rho(x)/\rho(z) \leq C$.*

Proof. From (3) we have

$$\frac{1}{\rho(z)} \leq \frac{1}{\rho(0)}\left(1 + \frac{|z|}{\rho(0)}\right)^{N_0}$$

and

$$\rho(x) \leq \rho(0)\left(1 + \frac{|x|}{\rho(0)}\right)^{\frac{N_0}{N_0+1}}.$$

Therefore, since $\rho(0)$ is a positive number we arrive to

$$\frac{\rho(x)}{\rho(z)} \leq C\frac{(1 + |x|)^{\frac{N_0}{N_0+1}}}{(1 + |z|)^{N_0}}.$$

Now, if $|x| \leq 2$ the quotient is bounded by a constant. Otherwise, for $|x| > 2$ and since $|x - z| \leq 1$ we have $|z| \geq |x|/2$ and also $1 + |x| \leq 3|x|/2$, so the ratio is bounded by a constant times $|x|^{-\frac{N_0^2}{N_0+1}}$ which is at most one.

□

Now we present the announced result.

Proposition 5. *Let ρ be a bounded function and $\theta \geq 1$. Then given $f \in \mathrm{SMO}_\rho^\theta$ and $\epsilon > 0$, there is a function $h \in \mathcal{C}_0^\infty$ such that $\|f - h\|_{\mathrm{BMO}_\rho^\theta} < \epsilon$.*

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Proof. First suppose that $f \in \mathrm{SMO}^\theta_\rho$ and bounded. Given $\epsilon > 0$ pick δ, λ and R as in the conclusion of Lemma 3. We may assume that $R > \lambda$ and $R \geq (C_\rho \epsilon)^{-1}$, where $C_\rho^{-1} = \|\rho\|_\infty$. Choose a function ψ_R smooth, $\psi_R = 1$ in $B(0, 2R)$, $0 \leq \psi_R \leq 1$, $\psi_R = 0$ in $B(0, 4R)^c$ with $|\nabla \psi_R| \leq 1/R$. We set $g = \psi_R f$, so g is a bounded function with compact support. Since

$$|g(x) - g(y)| \leq \|f\|_\infty |\psi_R(x) - \psi_R(y)| + |f(x) - f(y)|,$$

we have

$$\operatorname{Osc}^\theta_\rho(g, B(x_0, r)) \leq \|f\|_\infty \operatorname{Osc}^\theta_\rho(\psi_R, B(x_0, r)) + \operatorname{Osc}^\theta_\rho(f, B(x_0, r)). \tag{29}$$

Now we estimate the oscillations of ψ_R. By the Mean Value Theorem we have $|\psi_R(x) - \psi_R(y)| \leq |x - y| |\nabla \psi_R(\xi)|$, for some ξ, and hence

$$\operatorname{Osc}^\theta_\rho(\psi_R, B(x_0, r) \leq \frac{r}{(1 + rC_\rho)^\theta} \frac{1}{R},$$

with $C_\rho^{-1} = \|\rho\|_\infty$. Since $\theta \geq 1$ we have for any x_0 and r

$$\operatorname{Osc}^\theta_\rho(\psi_R, B(x_0, r) \leq \frac{1}{C_\rho R} \leq \epsilon,$$

due to our assumption on R. Plugging the last estimate in (29) we get

$$\operatorname{Osc}^\theta_\rho(g, B(x_0, r)) \leq (\|f\|_\infty + 1)\epsilon,$$

whenever $r < \delta$ or $r > \lambda$ and all x_0 or if $|x_0| > R$ and any $r > 0$. Therefore, for such balls we have

$$\operatorname{Osc}^\theta_\rho(f - g, B(x_0, r)) \leq (\|f\|_\infty + 2)\epsilon.$$

It remains to analyse balls $B(x_0, r)$ with $|x_0| < R$ and $\delta < r < \lambda$. However in that case, $B(x_0, r) \subset B(0, 2R)$ and hence $\psi_R = 1$ and so $f - g = 0$. That the function g actually belongs to SMO^θ_ρ, even it is not necessary, it is true. In fact conditions (ii) and (iii) follow from the integrability and its compact support respectively. Regarding condition (i), if we take limit for $r \to 0$ in (29), both terms go to zero since $f \in \mathrm{SMO}^\theta_\rho$ and ψ_R is an uniformly continuous function.

Next, we approximate $g \in \mathrm{SMO}^\theta_\rho$, bounded and with compact support, by a $\mathcal{C}^\infty_0$ function. To that end, take any non negative $\mathcal{C}^\infty_0$ function Φ with support contained in the unit ball such that $\int \Phi = 1$. Then $\Phi * g \in \mathcal{C}^\infty_0$ and hence $\Phi * g \in \mathrm{SMO}^\theta_\rho$.

Now we claim that there is a constant C such that for any ball $B(x_0, r)$,

$$\operatorname{Osc}^\theta_\rho(\Phi * g, B(x_0, r)) \leq C \sup_{|\xi| \leq 1} \operatorname{Osc}^\theta_\rho(g, B(x_0 - \xi, r)). \tag{30}$$

In fact,

$$\begin{aligned}
&\int_{B(x_0,r)} \int_{B(x_0,r)} |\Phi * g(x) - \Phi * g(y)| \, dx \, dy \\
&\qquad \leq \int \Phi(\xi) \left(\int_{B(x_0,r)} \int_{B(x_0,r)} |g(x - \xi) - g(y - \xi)| dx dy \right) d\xi \\
&\qquad\qquad \leq \sup_{|\xi| \leq 1} \int_{B(x_0 - \xi, r)} \int_{B(x_0 - \xi, r)} |g(x') - g(y')| dx' dy',
\end{aligned}$$

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where we used that Φ has its support contained in the unit ball and its integral is one. Now, multiplying both sides by $c_d^2 r^{-2d} (1+r/\rho(x_0))^{-\theta}$, where c_d is the measure of the unit ball, we arrive to

$$\mathrm{Osc}_\rho^\theta(\Phi * g, B(x_0,r)) \leq \sup_{|\xi|\leq 1} \frac{(1+r/\rho(x_0-\xi))^\theta}{(1+r/\rho(x_0))^\theta} \ \mathrm{Osc}_\rho^\theta(g, B(x_0-\xi,r)).$$

Since

$$\frac{1+r/\rho(x_0-\xi)}{1+r/\rho(x_0)} \leq 1 + \frac{\rho(x_0)}{\rho(x_0-\xi)},$$

using Lemma 5 we get (30). Observe that the claim holds for any smooth function Φ with integral one and whose support lies inside the unit ball. Therefore, taking a particular function ζ such that $0 \leq \zeta \leq 1$, $\operatorname{supp}\zeta \subset B(0,1)$ and $\int \zeta = 1$, we can apply inequality (30) to $\zeta_n(x) = n\zeta(nx)$, $n \geq 1$ with a uniform constant.

Now, let $h_n = \zeta_n * g$, $n \geq 1$. For a given $\epsilon > 0$, since $g \in \mathrm{SMO}_\rho^\theta$, pick δ, λ and $R > \lambda$ satisfying the conclusion of Lemma 3. Then for $r < \delta$ or $r > \lambda$ and any x_0, by (30) and the above remark, we have

$$\begin{aligned}\mathrm{Osc}_\rho^\theta(h_n - g, B(x_0,r)) &\leq \mathrm{Osc}_\rho^\theta(h_n, B(x_0,r)) + \mathrm{Osc}_\rho^\theta(g, B(x_0,r)) \\ &\leq 2C \sup_{|\xi|\leq 1} \mathrm{Osc}_\rho^\theta(g, B(x_0-\xi,r)) \\ &\leq 2C\,\epsilon.\end{aligned}$$

Also, assuming $R \geq 1$, for $|x_0| \geq 2R$ and any r we will have $|x_0 - \xi| \geq R$ and using again (30) we arrive to the same inequality. Again, it remains to consider oscillations for balls $B(x_0,r)$ with $|x_0| \leq 2R$ and $\delta < r < \lambda$. In that case

$$\begin{aligned}\mathrm{Osc}_\rho^\theta(h_n - g, B(x_0,r)) &\leq \frac{2}{|B(x_0,r)|}\int_{B(x_0,r)} |\zeta_n * g - g| \\ &\leq \frac{2}{c_d\delta^d}\int_{B(0,3R)} |\zeta_n * g - g|,\end{aligned}$$

and the last integral tends to zero for $n \to \infty$, since, being g bounded with compact support, the integrand goes to zero a.e and it is bounded by the integrable function $2g$. Altogether we have proved that for n large enough the function h_n belongs to $\mathcal{C}_0^\infty$ and satisfies

$$\|g - h_n\|_{\mathrm{BMO}_\rho^\theta} < C'\,\epsilon,$$

for some constant C'.

To conclude the proof of the Proposition we assume that $f \in \mathrm{SMO}_\rho^\theta$ but it is not necessary bounded.

In this case, we define for each positive N the bounded function

$$f_N(x) = \begin{cases} N & \text{if } f(x) > N, \\ f(x) & \text{if } |f(x)| \leq N, \\ -N & \text{if } f(x) < -N, \end{cases}$$

and we prove that $f_N \in \mathrm{SMO}_\rho^\theta$ and $\lim_{N\to\infty} f_N = f$ in the BMO_ρ^θ sense. Let us observe that if $g_N = \min\{f, N\}$, then $f_N = \max\{g_N, -N\} = -\min\{-g_N, N\}$. By Lemma 4

$$\mathrm{Osc}_\rho^\theta(f_N, B(x_0,r)) \leq 4\mathrm{Osc}_\rho^\theta(f, B(x_0,r)).$$

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Then given $\epsilon > 0$ let us consider for f the parameters δ, λ and R as given in Lemma 3. In this case we will have that

$$\operatorname{Osc}_\rho^\theta(f - f_N, B(x_0, r)) \le 5\epsilon$$

for any x_0 provided $r < \delta$ or $r > \lambda$ and for any r as long as $|x_0| > R$.

Next let us consider $B(x_0, r)$ such that $\delta < r < \lambda$ and $|x_0| \le R$. Thus

$$\operatorname{Osc}_\rho^\theta(f - f_N, B(x_0, r)) \le \frac{2}{|B(x_0, r)|} \int_{B(x_0,r)} |f - f_N| \le \frac{2}{c_d \delta^d} \int_{B(x_0,\lambda)} |f - f_N|.$$

But the last integral goes to zero when $N \to \infty$ as a consequence of Lebesgue Dominated Convergence Theorem.

□

As a consequence we have proved the following result.

Corollary 1. *Let ρ be a bounded critical radius function. Then* $\mathrm{SMO}_\rho^\infty = \mathrm{CMO}_\rho^\infty$.

We finish this section showing some examples of CMO_ρ^∞ functions. The first one shows that for any given critical radius function (not necessarily bounded) there exists a function $f \in \mathrm{CMO}_\rho^\infty$ such that $f \notin \mathrm{CMO}$. Moreover, our argument shows that the classical space BMO is always strictly contained in BMO_ρ^∞.

Let ρ a function satisfying (3). Using the second inequality in (3) and setting $\delta = N_0/N_0 + 1$, we have $\rho(x) \le C(1 + |x|)^\delta$ for some constant C only depending on ρ. Take $f(x) = |x|^\alpha + 1$ with $0 < \alpha < 1 - \delta$. We will show first that $f \in \mathrm{SMO}_\rho^\theta$ for θ such that $1 < \theta < (1 - \alpha)/\delta$. Consider a ball $B(x, r)$. If $|x| > 2r$,

$$\begin{aligned} \operatorname{Osc}_\rho^\theta(f, B(x, r)) &\le \left(1 + \frac{r}{\rho(x)}\right)^{-\theta} \fint_{B(x,r)} \fint_{B(x,r)} |f(y) - f(z)| dydz \\ &\le \left(1 + \frac{r}{\rho(x)}\right)^{-\theta} \fint_{B(x,r)} \fint_{B(x,r)} |z - y|(1 + |\xi|)^{\alpha-1} dydz \\ &\le C \left(1 + \frac{r}{\rho(x)}\right)^{-\theta} r(1 + |x|)^{\alpha-1}. \end{aligned} \tag{31}$$

Condition (i) is satisfied since, in particular, $\operatorname{Osc}(f, B(x, r)) \le Cr$. To check the remaining conditions recall that $\rho(x) \le C(1 + |x|)^\delta$. Hence,

$$\operatorname{Osc}_\rho^\theta(f, B(x, r)) \le \frac{C(1 + |x|)^{\alpha-1+\delta\theta}}{r^{\theta-1}}, \tag{32}$$

So that condition (ii) and condition (iii) follow since $\theta > 1$ and $\alpha - 1 + \delta\theta < 0$.

On the other hand, if $|x| \le 2r$,

$$\begin{aligned} \operatorname{Osc}_\rho^\theta(f, B(x, r)) &\le \left(1 + \frac{r}{\rho(x)}\right)^{-\theta} \fint_{B(x,r)} \fint_{B(x,r)} |f(y) - f(z)| dydz \\ &\le \left(1 + \frac{r}{\rho(x)}\right)^{-\theta} \fint_{B(0,3r)} \fint_{B(0,3r)} (|z|^\alpha + |y|^\alpha) \, dydz \\ &\le C \left(1 + \frac{r}{\rho(x)}\right)^{-\theta} r^\alpha. \end{aligned} \tag{33}$$

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Condition (i) is satisfied since, in particular, $\mathrm{Osc}(f, B(x,r)) \leq Cr^{\alpha}$. To check condition (ii) observe that

$$\mathrm{Osc}_{\rho}^{\theta}(f, B(x,r)) \leq C\frac{(1+|x|)^{\delta\theta}}{r^{\theta-\alpha}} \leq Cr^{\alpha+\delta\theta-\theta} \to 0$$

when $r \to \infty$, since $\theta > 1$ and $\alpha + \delta\theta - \theta < 0$. Condition (iii) follows from

$$\mathrm{Osc}_{\rho}^{\theta}(f, B(x,r)) \leq C\frac{(1+|x|)^{\delta\theta}}{r^{\theta-\alpha}} \leq C\frac{r^{\alpha}(1+|x|)^{\delta\theta}}{|x|^{\theta}},$$

since $\theta \geq 1$. In conclusion, we have shown that the function $f(x) = |x|^{\alpha} + 1$ with $0 < \alpha < 1 - \delta$ belongs to $\mathrm{SMO}_{\rho}^{\infty}$.

Now, if ρ is a bounded function, then $f \in \mathrm{CMO}_{\rho}^{\infty}$ by Corollary 1. If not, we are going to show that we can approximate f by a $\mathcal{C}_0^{\infty}$ function. Let $\varepsilon > 0$. we can still apply Lemma 3 to obtain δ, λ and R such that $\mathrm{Osc}_{\rho}^{\theta}(f, B(x,r)) < \varepsilon$ for any x provided $r < \delta$ or $r > \lambda$ and for any r as long as $|x| > R$. Also, we can assume $\delta < \lambda < R$. Defining $C_R = f(2R)$ and $g(x) = \min\{f(x), C_R\}$ we obtain, by Lemma 4,

$$\mathrm{Osc}_{\rho}^{\theta}(g, B(x,r)) \leq 2\mathrm{Osc}_{\rho}^{\theta}(f, B(x,r)) < 2\varepsilon$$

for any x provided $r < \delta$ or $r > \lambda$ and for any r as long as $|x| > R$. If $\delta \leq r \leq \lambda$ and $|x| \leq R$ we have $B(x,r) \subset B(0,2R)$. Therefore $f \equiv g$ and $\mathrm{Osc}_{\rho}^{\theta}(f-g, B(x,r)) = 0$. Finally, since $g \in \mathrm{SMO}_{\rho}^{\theta}$ is bounded an with compact support, we can proceed as in the last step of the proof of Proposition 5 to approximate it by a $\mathcal{C}_0^{\infty}$ function, proving that $f \in \mathrm{CMO}_{\rho}^{\infty}$. Clearly $f \notin \mathrm{CMO}$ since it is known that positive powers are not even in BMO.

We finish this section with an example to show that if ρ decreases at infinity as a negative power, $\mathrm{CMO}_{\rho}^{\infty}$ functions can increase more rapidly than those presented above.

Let $\rho(x) = (1+|x|)^{-k}$ for $k \in \mathbb{N}$ and $f(x) = |x|^{\alpha} + 1$. Since ρ is bounded we have $\mathrm{CMO}_{\rho}^{\infty} = \mathrm{SMO}_{\rho}^{\infty}$. Consider a ball $B = B(x,r)$. Suppose first that $|x| > 2r$. We can use the estimate for $\mathrm{Osc}_{\rho}^{\theta}(f, B(x,r))$ given in (31) to obtain

$$\mathrm{Osc}_{\rho}^{\theta}(f, B(x,r)) \leq C\frac{r(1+|x|)^{\alpha-1}}{(1+r(1+|x|)^{k})^{\theta}} \leq r^{1-\theta}(1+|x|)^{\alpha-1-k\theta}. \tag{34}$$

So, condition (i) holds as long as $\theta < 1$ and $\alpha - 1 - k\theta \leq 0$. Both requirements are fulfilled if we take $\alpha < k+1$. To check condition (ii) we write instead

$$\mathrm{Osc}_{\rho}^{\theta}(f, B(x,r)) \leq C\frac{r(1+|x|)^{\alpha-1}}{(1+r(1+|x|)^{k})^{\theta}} \leq r^{-\theta}(1+|x|)^{\alpha-k\theta}.$$

So, condition (ii) is satisfied as long as $\theta > 0$ and $\alpha \leq k\theta$. For condition (iii) we observe from (34) that it is verified as long as $\alpha < k\theta + 1$.

It remains to analyse the case $|x| \leq 2r$. From estimate for $\mathrm{Osc}_{\rho}^{\theta}(f, B(x,r))$ given in (33) we obtain

$$\mathrm{Osc}_{\rho}^{\theta}(f, B(x,r)) \leq C\left(1 + r(1+|x|)^{k}\right)^{-\theta} r^{\alpha}.$$

Condition (i) is satisfied as long as $\alpha > 0$ and $\theta \geq 0$. Finally we observe

$$\mathrm{Osc}_{\rho}^{\theta}(f, B(x,r)) \leq C(1+|x|)^{-k\theta} r^{\alpha-\theta}.$$

So, condition (ii) holds for $\theta > \alpha$ and condition (iii) holds for $\theta > 0$.

Altogether we have obtained that if $\alpha < k+1$:

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- condition (i) is true for $(\alpha-1)/k \le \theta_1 < 1$ and $\theta_1 > 0$,
- condition (ii) is true for $\theta_2 > \alpha$,
- condition (iii) is true for $\theta_3 > (\alpha-1)/k$ and $\theta_3 > 0$.

Due to the inequality

$$\mathrm{Osc}_\rho^\theta(f, B(x,r)) \le \mathrm{Osc}_\rho^{\theta'}(f, B(x,r))$$

when $\theta' \le \theta$, choosing $\theta = \max\{\theta_1, \theta_2, \theta_3\}$ we obtain that $f(x) = |x|^\alpha + 1$, for $\alpha < k+1$, belongs to $\mathrm{SMO}_\rho^\theta \subset \mathrm{SMO}_\rho^\infty = \mathrm{CMO}_\rho^\infty$ for $\rho(x) = (1+|x|)^{-k}$. In particular, for the Hermite operator $-\Delta + |x|^2$, since $\rho(x) = (1+|x|)^{-1}$, we have that $f(x) = |x|^\alpha + 1 \in \mathrm{CMO}_\rho^\infty$ for any $\alpha < 2$. Similar results can be obtained when the potential is a positive polynomial.

6. A final comment

Up to here we have dealt with commutators of linear operators. However there are relevant operators in the harmonic analysis related to the Schrödinger semigroup (as well as in the Laplacian case) which are not linear, such as maximal operators or square functions. A key idea to handle such operators is to think them as the norm, in an appropriate Banach space $\mathcal{B}$, of a linear operator mapping scalar functions in functions taking values in $\mathcal{B}$. For example, the maximal operator $W^*f(x) = \sup_{t>0} |W_t f(x)|$, where W_t is the semigroup e^{-tL}, can be seen as the norm in $\mathcal{B} = L^\infty((0,\infty))$ of a linear operator $\mathcal{W}$ mapping scalar functions into functions with values in $L^\infty((0,\infty))$. More precisely, setting $\mathcal{W}f(x) = (W_t f(x))_{t>0}$, we clearly have $W^*(x) = \|\mathcal{W}f(x)\|_{L^\infty(0,\infty)}$. Therefore boundedness of the maximal operator W^* in $L^p(w)$ means exactly that the linear operator $\mathcal{W}$ is bounded from $L^p(w)$ in $L^p_{\mathcal{B}}(w)$, since

$$\|W^*f\|^p_{L^p(w)} = \int_{\mathbb{R}^d} \|\mathcal{W}f(x)\|^p_{L^\infty(0,\infty)} w(x)dx = \|\mathcal{W}f\|^p_{L^p_{L^\infty(0,\infty)}(w)}.$$

Such linearization also allows us to define in a natural way commutators of W^* with a symbol b. In fact, being $\mathcal{W}$ a linear operator, commutators are defined by $\mathcal{W}_b f = b\mathcal{W}f - \mathcal{W}(bf)$. Notice that in the first term for each x we have multiplication of an scalar by a vector in $\mathcal{B}$, while in the second is just a product of numbers. Therefore, for the maximal operator commutators should be defined as

$$W_b^* f = \|\mathcal{W}_b f\|_{L^\infty(0,\infty)} = \sup_{t>0} |(W_t)_b f|.$$

Now, the question is whether Theorem 1 and Theorem 2 can be carried out to the context of vector valued operators. A first observation is that the same compactness criterion holds in $L^p_{\mathcal{B}}(wdx)$, just changing absolute values in the conditions by $\mathcal{B}$-norm. Also, as for boundedness, it is clear that compactness of $\mathcal{W}$ is equivalent to that of W^*.

Next we may introduce the corresponding family of vector valued operators ρ-Schrödinger-Calderón-Zygmund operators of type (s,δ), i.e. bounded from $L^{s'}$ into $L^{s',\infty}_{\mathcal{B}}$ and with kernels satisfying conditions (8) and (9), where absolute values on the left must be replaced by $\mathcal{B}$-norm.

Even we are not going to work out the details, it is more likely that proofs of boundedness and compactness for their commutators can be carried out obtaining the same symbols and weights stated in Theorem 1 and Theorem 2, just following the same steps with minor modifications.

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Consequently, such results could be applied, for instance, to maximal operators of the heat and Poisson semigroups related to the Schrödinger operator, taking $\mathcal{B} = L^\infty(0,\infty)$, as well as for the g-function introduced in [11], being $\mathcal{B} = L^2((0,\infty), dt/t)$ this time. The needed estimates to check that their vector valued versions are vector valued ρ-Schrödinger-Calderón-Zygmund operators of type (∞, δ) can be found in [11] and [16].

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Email address: bbongio@santafe-conicet.gov.ar

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Instituto de Matemática Aplicada del Litoral CONICET-UNL, and Facultad de Ingeniería Química, UNL. Colectora Ruta Nac. N 168, Paraje El Pozo.
Email address: harbour@santafe-conicet.gov.ar

Instituto de Matemática Aplicada del Litoral, CONICET-UNL. Colectora Ruta Nac. N 168, Paraje El Pozo. 3000 Santa Fe, Argentina.
Email address: pquijano@santafe-conicet.gov.ar

Instituto de Matemática Aplicada del Litoral, CONICET-UNL, and Facultad de Ingeniería Química, UNL. Colectora Ruta Nac. N 168, Paraje El Pozo. 3000 Santa Fe, Argentina.